

Double scaling limit for modified Jacobi-Anglesco polynomials

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Dedicated to the memory of Julius Borcea

Abstract. We consider multiple orthogonal polynomials with respect to two modified Jacobi weights on touching intervals $[a, 0]$ and $[0, 1]$, with $a < 0$, and study a transition that occurs at $a = -1$. The transition is studied in a double scaling limit, where we let the degree n of the polynomial tend to infinity while the parameter a tends to -1 at a rate of $O(n^{-1/2})$. We obtain a Mehler-Heine type asymptotic formula for the polynomials in this regime. The method used to analyze the problem is the steepest descent technique for Riemann-Hilbert problems. A key point in the analysis is the construction of a new local parametrix.

1. Introduction and statement of results

1.1. Introduction

Multiple orthogonal polynomials are a generalization of orthogonal polynomials that originated in works on Hermite-Padé rational approximation problems, but recently found other applications in random matrix theory and related probabilistic models.

In the approximation theory literature two main classes of multiple orthogonal polynomials were identified for which detailed asymptotic results are available. These are the Angelesco systems and the Nikishin systems. In an Angelesco system [1] the multiple orthogonality is defined on disjoint intervals, while in a Nikishin system [32] the orthogonality is on the same interval with orthogonality measures that are related to each other via an intricate hierarchical structure.

A main stimulus for the asymptotic analysis of orthogonal polynomials was given by the formulation of a 2×2 matrix valued Riemann-Hilbert problem for orthogonal polynomials by Fokas, Its and Kitaev [21] and the subsequent application of the powerful Deift-Zhou steepest descent technique to this Riemann-Hilbert problem in [15, 16] and many later papers.

A Riemann-Hilbert problem for multiple orthogonal polynomials was formulated by Van Assche, Geronimo and Kuijlaars [41]. The Riemann-Hilbert problem is of size $(r+1) \times (r+1)$, where r is the number of orthogonality weights for the multiple orthogonal polynomials. The Riemann-Hilbert formulation was already

used in several papers, see e.g. [4, 6, 7, 9, 10, 11, 20, 27, 30, 31] for the asymptotic analysis of multiple orthogonal polynomials and their associated multiple orthogonal polynomial ensembles [25, 26].

In this paper we consider Angelesco systems on two touching intervals $[a, 0]$ and $[0, 1]$ with $a < -1$. Our interest is in the special behavior at 0 that takes place near a critical value of a . A prime example for this situation is given by the Jacobi-Angelesco weights

$$\begin{aligned} w_1(x) &= |x - a|^\alpha |x|^\beta |x - 1|^\gamma, & x \in (a, 0), \\ w_2(x) &= |x - a|^\alpha |x|^\beta |x - 1|^\gamma, & x \in (0, 1), \end{aligned} \quad (1.1)$$

with $\alpha, \beta, \gamma > -1$, which were first studied by Kaliaguine [23, 24]. The associated multiple orthogonal polynomials are among the classical multiple orthogonal polynomials [5] and as such have a number of very special properties. There is e.g. a raising operator which gives rise to a Rodrigues-type formula and a third order linear differential equation as well as an explicit four term recurrence relation for the diagonal case Jacobi-Angelesco multiple orthogonal polynomials, see [2, 24, 35, 36, 37, 38, 40].

1.2. Modified Jacobi-Angelesco weights

We generalize the system (1.1) by considering more general modified Jacobi weights on the two intervals $(a, 0)$ and $(0, 1)$. We will use the following weights w_1 and w_2 throughout this paper.

Definition 1.1. Let $a < 0$, $\alpha, \beta, \gamma > -1$ and define

$$\Delta_1 = [a, 0], \quad \Delta_2 = [0, 1].$$

For $j = 1, 2$, let h_j be strictly positive on Δ_j with an analytic continuation to a neighborhood of Δ_j in the complex plane. Then we define

$$\begin{aligned} w_1(x) &= (x - a)^\alpha |x|^\beta h_1(x), & x \in \Delta_1, \\ w_2(x) &= x^\beta (1 - x)^\gamma h_2(x), & x \in \Delta_2. \end{aligned} \quad (1.2)$$

When appropriate we set $w_j(x) \equiv 0$ for $x \in \mathbb{R} \setminus \Delta_j$.

The definition of the multiple orthogonal polynomial (of type II) with respect to the weights (1.2) is as follows.

Definition 1.2. Given a multi-index $(n_1, n_2) \in \mathbb{N}^2$ the multiple orthogonal polynomial is defined as the unique monic polynomial P_{n_1, n_2} of degree $n_1 + n_2$ such that

$$\int_{\Delta_j} P_{n_1, n_2}(x) x^k w_j(x) dx = 0, \quad \text{for } k = 0, \dots, n_j - 1, \quad (1.3)$$

for $j = 1, 2$.

Since we are dealing with an Angelesco system of weights [1] the polynomial P_{n_1, n_2} indeed exists and is uniquely characterized by (1.3). It is also known that all the zeros of P_{n_1, n_2} are real and simple with n_1 zeros in $(a, 0)$ and n_2 zeros in

$(0, 1)$, see e.g. [39]. For the definition of the multiple orthogonal polynomials of type I we also refer to [39].

1.3. The phase transition

We consider in this paper the diagonal case

$$n_1 = n_2 = n.$$

It is known that the zeros of the multiple orthogonal polynomial $P_{n,n}$ have a weak limit as $n \rightarrow \infty$, which only depends on the parameter $a < 0$. The limiting zero distribution can be characterized as the solution to a vector equilibrium problem for two measures [22, 32].

Define the logarithmic energy $I(\nu)$ of a measure ν as

$$I(\nu) := \iint \log \frac{1}{|x - y|} d\nu(x) d\nu(y), \quad (1.4)$$

and the mutual logarithmic energy $I(\nu, \mu)$ of two measures ν and μ as

$$I(\nu, \mu) := \iint \log \frac{1}{|x - y|} d\nu(x) d\mu(y). \quad (1.5)$$

Then the vector equilibrium problem is defined as follows.

Definition 1.3. The vector equilibrium problem asks to minimize the energy functional

$$E(\nu_1, \nu_2) := I(\nu_1) + I(\nu_1, \nu_2) + I(\nu_2) \quad (1.6)$$

among positive measures ν_1 and ν_2 with $\text{supp}(\nu_1) \subset [a, 0]$, $\text{supp}(\nu_2) \subset [0, 1]$ and $\int d\nu_1 = \frac{1}{2}$, $\int d\nu_2 = \frac{1}{2}$

One may interpret this energy functional as the energy resulting from two conductors $[a, 0]$ and $[0, 1]$ with each an equal amount of charged particles. Particles on the same conductor repel each other, such that the resulting electrostatic force is proportional to the inverse of the distance between the two particles, which accounts for the terms $I(\nu_1)$ and $I(\nu_2)$. Additionally, particles on different conductors also repel each other, but with only half the strength. This leads to the term $I(\nu_1, \nu_2)$ in (1.6). This kind of interaction is known as Angelesco-type interaction, see [3].

The minimizers ν_1 and ν_2 for the Angelesco equilibrium problem are called the equilibrium measures. They exist, are unique and are absolutely continuous with respect to the Lebesgue measure. It is due to Kaliaguine [23] that the endpoints of the supports of the equilibrium measures are given by $a, 0, 1$ and a fourth point b :

$$b = \frac{(a + 1)^3}{9(a^2 - a + 1)} \quad (1.7)$$

such that

$$\begin{aligned} \text{supp } \nu_1 &= [a, b] \subset [a, 0], & \text{supp } \nu_2 &= [0, 1], & \text{if } a &\leq -1, \\ \text{supp } \nu_1 &= [a, 0], & \text{supp } \nu_2 &= [b, 1] \subset [0, 1], & \text{if } a &\geq -1. \end{aligned} \quad (1.8)$$

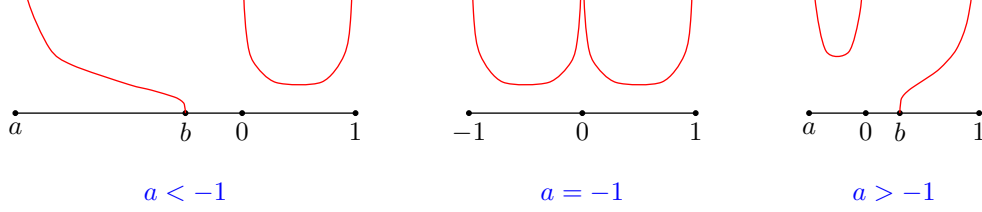


FIGURE 1. A sketch of the equilibrium densities in the three cases $a < -1$, $a = -1$, and $-1 < a < 0$. For a clearer picture the size of the gap between 0 and b has been exaggerated. In a true plot b would be much closer to 0 as the gap is less than one ninth of the length of the larger interval.

We see here the pushing effect: the charge on the smaller interval pushes away the charge on the larger interval, thereby creating a gap in the support. The gap disappears in the symmetric case $a = -1$ where we have $b = 0$.

The density of the equilibrium measures blows up as an inverse square root at the endpoints $a, 0$, and 1 of its supports. These are the so-called hard edges. For $a \neq -1$ there is a soft edge at b , where the equilibrium density vanishes like a square root. In the symmetric case $a = -1$, where both intervals have equal size, both measures have full supports and the densities behave like inverse cube roots at 0 . We may call 0 in this case a Kaliaguine point after [23]. A sketch of the densities in the three cases is given in Figure 1.

It is this interior soft-to-hard edge transition as a varies around -1 that will give rise to a new critical behavior of the multiple orthogonal polynomials around 0 that we wish to describe in this paper.

1.4. Main result

The main result of this paper is a Mehler-Heine type asymptotic formula for the multiple orthogonal polynomial $P_{n,n}(z)$ near $z = 0$, with the parameter a near -1 . We use $P_{n,n}(z; a)$ to denote the dependence on a .

Theorem 1.4. *For $a < 0$ close enough to -1 , let $P_{n,n}(z; a)$ be the multiple orthogonal polynomial with respect to the weights (1.2) and the multi-index (n, n) . Let $\tau \in \mathbb{R}$ and*

$$a_n = -1 + \frac{\sqrt{2}\tau}{n^{1/2}}. \quad (1.9)$$

Then we have for every $z \in \mathbb{C}$,

$$P_{n,n}\left(\frac{z}{\sqrt{2}n^{3/2}}; a_n\right) = (-1)^n C_n Q(z; \tau) \left(1 + \mathcal{O}(n^{-1/6})\right) \quad (1.10)$$

where C_n is a positive constant and

$$Q(z; \tau) = i \int_{\Gamma_0} t^{-\beta-1} \exp\left(-\frac{z^2}{2t^2} - \frac{\tau z}{t} + t\right) dt, \quad z \in \mathbb{C}, \quad (1.11)$$

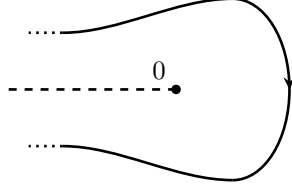


FIGURE 2. The contour Γ_0 appearing in the Mehler-Heine formula (1.10). The dashed line denotes the cut of $t^{-\beta-1}$.

where the contour Γ_0 is shown in of Figure 2. The convergence in (1.10) is uniform for z in compact subsets of \mathbb{C} .

The function Q in (1.11) is an entire solution of the third order differential equation

$$z^2 Q'''(z) + 2(\beta + 1)zQ''(z) + (\beta^2 + \beta - \tau z)Q'(z) + (z - \tau\beta)Q(z) = 0 \quad (1.12)$$

The differential equation has a regular singular point at $z = 0$, with associated Frobenius indices equal to 0 , $-\beta$ and $-\beta+1$. For $\beta > -1$, there is a one-dimensional space of entire solutions to (1.12), unless $\beta = 0$ in which case this space is two-dimensional. In case $\beta \neq 0$ we may characterize Q as the unique entire solution of (1.12) satisfying

$$Q(0) = \frac{2\pi}{\Gamma(\beta + 1)}. \quad (1.13)$$

The constant C_n in (1.10) is given by

$$C_n = \frac{e^{c_1+c_2}}{\sqrt{3\pi}} \frac{2^\beta}{3^{\frac{\alpha+2\beta+\gamma}{2}}} e^{-\tau^2 n^{\beta+\frac{1}{2}}} e^{-\sqrt{2}\tau n^{\frac{1}{2}}} \left(\frac{4}{27}\right)^n \quad (1.14)$$

where c_1 and c_2 are positive constants defined in (4.44) below. They are determined by the analytic factors h_1 and h_2 in the weights (1.2), and are independent of n . For simple analytic factors h_1 and h_2 one can evaluate c_1 and c_2 explicitly. For example, if h_k is a constant function, then $c_k = 0$.

Remark 1.5. In the case $\tau = 0$ the function $Q(z; \tau)$ from (1.11) can be written as a generalized hypergeometric function

$$Q(z; 0) = \frac{2\pi}{\Gamma(\beta + 1)} {}_0F_2\left(-; \frac{\beta + 1}{2}, \frac{\beta + 2}{2}; -\frac{z^2}{8}\right).$$

This function was already found by Sorokin [34] in a Mehler-Heine formula for certain multiple orthogonal polynomials of Laguerre-type. More recently, it was obtained for Jacobi-Anglesco multiple orthogonal polynomials by Tulyakov [37] and Takata [36], who both prove Theorem 1.4 for the case $\tau = 0$ and weights (1.2) with $h_1 \equiv 1$, $h_2 \equiv 1$.

Remark 1.6. In [40] an explicit formula for the Jacobi-Angelesco polynomial $P_{n,n}^{(\alpha,\beta,\gamma)}$ is given, that is, the multiple orthogonal polynomial with weights (1.1), namely

$$\begin{aligned} & \binom{3n+\alpha+\beta+\gamma}{n} P_{n,n}^{(\alpha,\beta,\gamma)}(z;a) \\ &= \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n+\alpha}{k} \binom{n+\beta}{j} \binom{n+\gamma}{n-k-j} (z-a)^{n-k} z^{n-j} (z-1)^{k+j}. \end{aligned} \quad (1.15)$$

Applying Stirling's approximation formula to the binomial coefficients, one can then derive that with a_n given by (1.9),

$$\begin{aligned} P_{n,n}(0;a_n) &= (-1)^n \frac{2\pi}{\Gamma(1+\beta)} \frac{1}{\sqrt{3\pi}} \left(\frac{2}{3}\right)^{\alpha+\beta+\gamma} e^{-\tau^2} \\ &\quad \times n^{\beta+\frac{1}{2}} e^{-\sqrt{2}\tau n^{\frac{1}{2}}} \left(\frac{4}{27}\right)^n \left(1 + \mathcal{O}\left(n^{-\frac{1}{2}}\right)\right) \end{aligned} \quad (1.16)$$

which is consistent with (1.14), since in this case one can evaluate c_1 and c_2 to be

$$c_1 = \gamma(\log 2 - \tfrac{1}{2} \log 3), \quad c_2 = \alpha(\log 2 - \tfrac{1}{2} \log 3). \quad (1.17)$$

1.5. Overview of the rest of the paper

We use two main tools to prove the Theorem 1.4, namely Riemann-Hilbert (RH) problems and modified equilibrium problems. These will be discussed in the next two sections.

The RH problems are of size 3×3 . We first discuss the RH problem for the multiple orthogonal polynomial P_{n_1,n_2} with the modified Jacobi weights. The steepest descent analysis of the paper will lead to a local parametrix that is built out of a local model RH problem, that is discussed in detail in Section 2.2. This model RH problem is new, although it is related to another model RH problem studied recently in a different connection [28].

The modified equilibrium problem is related to a Riemann surface in Section 3. The same Riemann surface will also play a role in the construction of the outer parametrix in the steepest descent analysis.

Section 4 is the bulk of the paper. It contains the steepest descent analysis of the RH problem for multiple orthogonal polynomials. It follows the usual steps in such an analysis as e.g. done in [14, 15, 29]. In a first transformation we use the g -functions coming from the modified equilibrium problem to normalize the RH problem at infinity. The next transformation is the opening of lenses. Then we construct outer and local parametrices that are used in the next transformation. After this transformation one typically arrives at a RH problem that is normalized at infinity, and for which the jump matrices all tend to the identity matrix as $n \rightarrow \infty$. It is a curious fact that this does not happen in the present paper. The jump matrix on a circle around 0 will take the form

$$I + Z_n(z) + \mathcal{O}(n^{-1/6})$$

where $Z_n(z)$ is bounded on the circle, but it does not tend to 0 as $n \rightarrow \infty$. We can resolve this problem by making another transformation, where we use the special structure of the matrices $Z_n(z)$. This extra step in the steepest descent analysis is also used in the recent papers [19, 28], which makes it reasonable to suspect that the need for such an extra step is a more common phenomenon in the steepest descent analysis of larger size RH problems in a critical situation.

The proof of Theorem 1.4 is given in the final section 6. Here we unravel all the previous transformations, and we pay special attention to the behavior around 0.

In a forthcoming paper we plan to analyze the determinantal point process that is associated with the modified Jacobi-Anglesco weights. This is an example of a multiple orthogonal polynomial ensemble [25] where half of the particles are on $[a, 0]$ and the other half are on $[0, 1]$. There is again a critical behavior at 0 as a varies around -1 , and we will find a new family of limiting correlation kernels in this setting that are also related to the solution of the local model RH problem.

2. First tool: RH problems

2.1. The Riemann-Hilbert problem

The multiple orthogonal polynomial (1.3) are characterized in terms of a 3×3 matrix valued Riemann-Hilbert problem (RH problem) due to [41]. We use the RH problem for the asymptotic analysis to derive our results.

We work with the modified Jacobi weights w_1 and w_2 (1.2) and we take a general multi-index (n_1, n_2) . The RH problem then asks for a function $Y : \mathbb{C} \setminus [a, 1] \rightarrow \mathbb{C}^{3 \times 3}$ such that

- Y is analytic on $\mathbb{C} \setminus [a, 1]$,
- Y has continuous boundary values Y_{\pm} on $(a, 0)$ and $(0, 1)$ satisfying a jump relation $Y_+ = Y_- J_Y$ with jump matrix

$$J_Y(x) = \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (a, 0) \cup (0, 1), \quad (2.1)$$

where it is understood that $w_1(x) \equiv 0$ on $(0, 1)$ and $w_2(x) \equiv 0$ on $(-a, 0)$,

- Y has the asymptotic behavior

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^{n_1+n_2} & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix} \quad (2.2)$$

as $z \rightarrow \infty$, and

- Y has the following behavior at the endpoints of the intervals

$$Y(z) = \mathcal{O} \begin{pmatrix} 1 & \epsilon(z) & 1 \\ 1 & \epsilon(z) & 1 \\ 1 & \epsilon(z) & 1 \end{pmatrix}, \text{ as } z \rightarrow a,$$

$$\text{where } \epsilon(z) = \begin{cases} |z - a|^\alpha & \text{if } \alpha < 0, \\ \log |z - a| & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha > 0 \end{cases} \quad (2.3)$$

$$Y(z) = \mathcal{O} \begin{pmatrix} 1 & 1 & \epsilon(z) \\ 1 & 1 & \epsilon(z) \\ 1 & 1 & \epsilon(z) \end{pmatrix}, \text{ as } z \rightarrow 1,$$

$$\text{where } \epsilon(z) = \begin{cases} |z - 1|^\gamma & \text{if } \gamma < 0, \\ \log |z - 1| & \text{if } \gamma = 0, \\ 1 & \text{if } \gamma > 0 \end{cases} \quad (2.4)$$

$$Y(z) = \mathcal{O} \begin{pmatrix} 1 & \epsilon(z) & \epsilon(z) \\ 1 & \epsilon(z) & \epsilon(z) \\ 1 & \epsilon(z) & \epsilon(z) \end{pmatrix}, \text{ as } z \rightarrow 0,$$

$$\text{where } \epsilon(z) = \begin{cases} |z|^\beta & \text{if } \beta < 0, \\ \log |z| & \text{if } \beta = 0, \\ 1 & \text{if } \beta > 0, \end{cases} \quad (2.5)$$

where the \mathcal{O} is taken entry-wise.

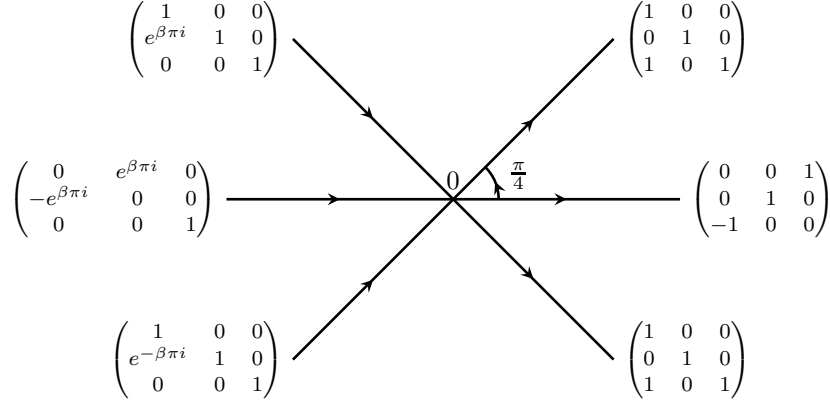
As in [14, 41] one can show that there is a unique solution of the RH problem, see also [29] for the role of the endpoint conditions (2.3), (2.4) and (2.5). The first column contains the multiple orthogonal polynomials of type II with respect to multi-indices (n_1, n_2) , $(n_1 - 1, n_2)$ and $(n_1, n_2 - 1)$ and the other columns contain Cauchy transforms of the polynomials times the weights. Indeed, the solution is equal to

$$Y(z) = \begin{pmatrix} P_{n_1, n_2}(z) & \frac{1}{2\pi i} \int_a^0 \frac{P_{n_1, n_2}(x)w_1(x)}{x-z} dx & \frac{1}{2\pi i} \int_0^1 \frac{P_{n_1, n_2}(x)w_2(x)}{x-z} dx \\ d_1 P_{n_1-1, n_2}(z) & \frac{d_1}{2\pi i} \int_a^0 \frac{P_{n_1-1, n_2}(x)w_1(x)}{x-z} dx & \frac{d_1}{2\pi i} \int_0^1 \frac{P_{n_1-1, n_2}(x)w_2(x)}{x-z} dx \\ d_2 P_{n_1, n_2-1}(z) & \frac{d_2}{2\pi i} \int_a^0 \frac{P_{n_1, n_2-1}(x)w_1(x)}{x-z} dx & \frac{d_2}{2\pi i} \int_0^1 \frac{P_{n_1, n_2-1}(x)w_2(x)}{x-z} dx \end{pmatrix} \quad (2.6)$$

for certain non-zero constants d_1 and d_2 . The inverse matrix Y^{-1} contains multiple orthogonal polynomials of type I.

We apply the Deift-Zhou steepest descent analysis to the RH problem for Y with $n_1 = n_2 = n$ in the limit where $n \rightarrow \infty$ and $a = a_n = -1 + \frac{\sqrt{2}\tau}{n^{1/2}} \rightarrow -1$. Via a number of transformations

$$Y \mapsto T \mapsto S \mapsto R$$

FIGURE 3. The contour Σ_Ψ and the jump matrices of Ψ .

we arrive at a matrix valued function R that tends to the identity matrix as $n \rightarrow \infty$.

Particularly relevant references on the steepest descent method for this paper are [10, 15, 29], see also [11].

2.2. The local model RH problem

At a crucial step in the steepest descent analysis we need to do a local analysis at the point 0. We have to construct there a local parametrix that will be built out of certain special functions. In non-critical situations this can be done with Bessel functions of order β , but in the critical regime that we are interested in we need functions that satisfy a third order linear differential equation. They are combined in a 3×3 matrix valued RH problem that we call the local model RH problem and that we describe next.

In the local model RH problem we are looking for a 3×3 matrix valued function Ψ that depends on two parameters $\beta > -1$ and $\tau \in \mathbb{R}$. Since β is considered fixed, we do not emphasize the dependence on β . We may write $\Psi(z; \tau)$ to emphasize the dependence on τ . Then Ψ should satisfy the following.

- Ψ is defined and analytic in $\mathbb{C} \setminus \Sigma_\Psi$ where Σ_Ψ is the contour consisting of the six oriented rays through the origin as shown in Figure 3,
- Ψ has continuous boundary values on Σ_Ψ that satisfy the jump condition

$$\Psi_+(z) = \Psi_-(z)J_\Psi(z) \quad z \in \Sigma_\Psi,$$

where the jump matrices J_Ψ are also given in Figure 3.

Thus the parameter β appears in the jump condition.

The dependence on τ is in the asymptotic condition as $z \rightarrow \infty$. We write throughout this paper

$$\omega = e^{2\pi i/3}.$$

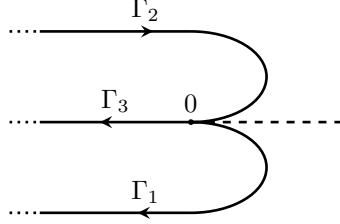


FIGURE 4. The contours Γ_1, Γ_2 and Γ_3 in the t -plane. The dashed line denotes the cut of $t^{-\beta-3}$.

- As $z \rightarrow \infty$ with $\pm \operatorname{Im} z > 0$, we have

$$\Psi(z) = \sqrt{\frac{2\pi}{3}} e^{\frac{\tau^2}{6} z^{\frac{\beta}{3}}} \begin{pmatrix} z^{\frac{1}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-\frac{1}{3}} \end{pmatrix} \Omega_{\pm} \left(I + \mathcal{O}\left(z^{-\frac{1}{3}}\right) \right) B_{\pm} e^{\Theta(z;\tau)}, \quad (2.7)$$

where Ω_{\pm} , B_{\pm} and $\Theta(z;\tau)$ are defined by

$$\begin{aligned} \Omega_+ &:= \begin{pmatrix} -\omega^2 & 1 & \omega \\ 1 & -1 & -1 \\ -\omega & 1 & \omega^2 \end{pmatrix}, & B_+ &:= \begin{pmatrix} e^{\frac{\beta\pi i}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\frac{\beta\pi i}{3}} \end{pmatrix}, \\ \Omega_- &:= \begin{pmatrix} \omega & 1 & \omega^2 \\ -1 & -1 & -1 \\ \omega^2 & 1 & \omega \end{pmatrix}, & B_- &:= \begin{pmatrix} e^{-\frac{\beta\pi i}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{\beta\pi i}{3}} \end{pmatrix}, \end{aligned} \quad (2.8)$$

and

$$\Theta(z;\tau) := \begin{cases} \operatorname{diag}(\theta_1(z;\tau), \theta_3(z;\tau), \theta_2(z;\tau)) & \text{for } \operatorname{Im} z > 0, \\ \operatorname{diag}(\theta_2(z;\tau), \theta_3(z;\tau), \theta_1(z;\tau)) & \text{for } \operatorname{Im} z < 0, \end{cases} \quad (2.9)$$

and the θ_k are defined by

$$\theta_k(z;\tau) := -\frac{3}{2}\omega^k z^{\frac{2}{3}} - \tau\omega^{2k} z^{\frac{1}{3}} \quad \text{for } k = 1, 2, 3. \quad (2.10)$$

The expansion (2.7) for $\Psi(z)$ as $z \rightarrow \infty$ is valid uniformly for τ in a bounded set.

We construct $\Psi(z;\tau)$ out of solutions of the third order linear differential equation

$$zq'''(z) - \beta q''(z) - \tau q'(z) + q(z) = 0. \quad (2.11)$$

Note that this is not the same differential equation as (1.12). However, the two are related, since if q satisfies (2.11) then

$$Q(z) = z^{-\beta} q''(z)$$

satisfies (1.12).

The differential equation (2.11) has solutions in the form of contour integrals

$$q(z) = \int_{\Gamma} t^{-\beta-3} e^{\frac{\tau}{t} - \frac{1}{2it^2} + zt} dt, \quad (2.12)$$

where Γ is an appropriate contour so that the integrand vanishes at the endpoints of the contour Γ . Define three contours Γ_1, Γ_2 and Γ_3 as in Figure 4, and define for z with $\operatorname{Re} z > 0$

$$q_j(z) = \int_{\Gamma_j} t^{-\beta-3} e^{\frac{\tau}{t} - \frac{1}{2it^2} + zt} dt, \quad j = 1, 2, 3, \quad (2.13)$$

where we choose the branch of $t^{-\beta-3}$ with a cut on the positive real axis, i.e.,

$$t^{-\beta-3} = |t|^{-\beta-3} e^{(-\beta-3)i \arg t}, \quad 0 < \arg t < 2\pi.$$

The integrals (2.13) only converge for z with $\operatorname{Re} z > 0$, but the functions q_j can be continued analytically using contour deformations. Branch points for the q_j -functions are 0 and ∞ and we take the analytic continuation to $\mathbb{C} \setminus (-\infty, 0]$, thus with a branch cut on the negative real axis.

Definition 2.1. Define Ψ in the upper half plane by

$$\Psi = \begin{cases} \begin{pmatrix} e^{2\beta\pi i} q_1 & e^{\beta\pi i} q_3 & q_2 \\ e^{2\beta\pi i} q'_1 & e^{\beta\pi i} q'_3 & q'_2 \\ e^{2\beta\pi i} q''_1 & e^{\beta\pi i} q''_3 & q''_2 \end{pmatrix} & 0 < \arg z < \frac{\pi}{4}, \\ \begin{pmatrix} e^{2\beta\pi i} q_1 + q_2 & e^{\beta\pi i} q_3 & q_2 \\ e^{2\beta\pi i} q'_1 + q'_2 & e^{\beta\pi i} q'_3 & q'_2 \\ e^{2\beta\pi i} q''_1 + q''_2 & e^{\beta\pi i} q''_3 & q''_2 \end{pmatrix}, & \frac{\pi}{4} < \arg z < \frac{3\pi}{4} \\ \begin{pmatrix} e^{2\beta\pi i} q_1 + q_2 - e^{2\beta\pi i} q_3 & e^{\beta\pi i} q_3 & q_2 \\ e^{2\beta\pi i} q'_1 + q'_2 - e^{2\beta\pi i} q'_3 & e^{\beta\pi i} q'_3 & q'_2 \\ e^{2\beta\pi i} q''_1 + q''_2 - e^{2\beta\pi i} q''_3 & e^{\beta\pi i} q''_3 & q''_2 \end{pmatrix}, & \frac{3\pi}{4} < \arg z < \pi, \end{cases} \quad (2.14)$$

and in the lower half plane by

$$\Psi = \begin{cases} \begin{pmatrix} q_2 & e^{\beta\pi i} q_3 & -e^{2\beta\pi i} q_1 \\ q'_2 & e^{\beta\pi i} q'_3 & -e^{2\beta\pi i} q'_1 \\ q''_2 & e^{\beta\pi i} q''_3 & -e^{2\beta\pi i} q''_1 \end{pmatrix}, & -\frac{\pi}{4} < \arg z < 0, \\ \begin{pmatrix} q_2 + e^{2\beta\pi i} q_1 & e^{\beta\pi i} q_3 & -e^{2\beta\pi i} q_1 \\ q'_2 + e^{2\beta\pi i} q'_1 & e^{\beta\pi i} q'_3 & -e^{2\beta\pi i} q'_1 \\ q''_2 + e^{2\beta\pi i} q''_1 & e^{\beta\pi i} q''_3 & -e^{2\beta\pi i} q''_1 \end{pmatrix}, & -\frac{3\pi}{4} < \arg z < -\frac{\pi}{4} \\ \begin{pmatrix} e^{2\beta\pi i} q_1 + q_2 + q_3 & e^{\beta\pi i} q_3 & -e^{2\beta\pi i} q_1 \\ e^{2\beta\pi i} q'_1 + q'_2 + q'_3 & e^{\beta\pi i} q'_3 & -e^{2\beta\pi i} q'_1 \\ e^{2\beta\pi i} q''_1 + q''_2 + q''_3 & e^{\beta\pi i} q''_3 & -e^{2\beta\pi i} q''_1 \end{pmatrix}, & -\pi < \arg z < -\frac{3\pi}{4}. \end{cases} \quad (2.15)$$

It is then an easy exercise to check that Ψ indeed satisfies the required jumps $\Psi_+ = \Psi_- J_{\Psi}$ on the rays $\arg z = 0, \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$. For the jump on the negative real

axis however, we have to take into consideration the behavior of the functions $q_j(z)$ as z circles around 0. Using contour deformations one can show that for $z < 0$:

$$\begin{pmatrix} q_{1,+}(z) \\ q_{2,+}(z) \\ q_{3,+}(z) \end{pmatrix} = \begin{pmatrix} 1 + e^{2\beta\pi i} & 1 & 0 \\ -e^{2\beta\pi i} & 0 & 0 \\ e^{2\beta\pi i} & 1 & 1 \end{pmatrix} \begin{pmatrix} q_{1,-}(z) \\ q_{2,-}(z) \\ q_{3,-}(z) \end{pmatrix}. \quad (2.16)$$

The jump of Ψ on the negative real axis follows from this in a straightforward way.

As for the asymptotic behavior, we have

Proposition 2.2. *The function Ψ defined in (2.14) and (2.15) satisfies the asymptotic condition (2.7).*

Proof. This follows from a classical steepest descent analysis applied to the contour integral representations for the q_j (2.13). Define the phase function $\theta(t; z, \tau)$ by

$$\theta(t; z, \tau) := \frac{\tau}{t} - \frac{1}{2t^2} + zt. \quad (2.17)$$

The main contributions in the integrals occurs around the saddles $t = t_k = t_k(z; \tau)$ of θ , which are the solutions to $\theta'(t) = 0$:

$$t_k(z; \tau) = -\omega^k z^{-\frac{1}{3}} - \frac{\tau}{3} \omega^{2k} z^{-\frac{2}{3}} + \mathcal{O}\left(z^{-\frac{4}{3}}\right), \quad k = 1, 2, 3. \quad (2.18)$$

The \mathcal{O} -term here is uniform for τ in compacta. The critical values are given by

$$\theta(t_k) = \theta(t_k; z, \tau) = -\frac{3}{2} \omega^k z^{\frac{2}{3}} - \tau \omega^{2k} z^{\frac{1}{3}} + \frac{\tau^2}{6} + \mathcal{O}\left(z^{-\frac{1}{3}}\right). \quad (2.19)$$

We also need the second derivative of θ in the saddle points:

$$\theta''(t_k) = -3\omega^{2k} z^{\frac{4}{3}} + \mathcal{O}(z). \quad (2.20)$$

Through each saddle point t_k there is a steepest descent path Υ_k . This is a path such that $\text{Im } \theta(t) = \text{Im } \theta(t_k)$ for all $t \in \Upsilon_k$. Let $\alpha_k, |\alpha_k| = 1$ be the tangent direction of Υ_k in t_k . The steepest descent method then yields

$$\int_{\Upsilon_k} t^{-\beta-3} e^{\theta(t)} dt = \alpha_k \sqrt{\frac{2\pi}{-\theta''(t_k(z)) \alpha_k^2}} t_k(z)^{-\beta-3} e^{\theta(t_k(z))} \left(1 + \mathcal{O}(z^{-1/3})\right). \quad (2.21)$$

The fact that Υ_k is a steepest descent path guarantees that $\theta''(t_k(z)) \alpha_k^2$ is negative. Substituting (2.18), (2.19) and (2.20) we find

$$\begin{aligned} & \int_{\Upsilon_k} t^{-\beta-1} e^{\theta(t)} dt \\ &= \pm \sqrt{\frac{2\pi}{3}} e^{\frac{\tau^2}{6}} \omega^{2k} z^{-\frac{2}{3}} (-\omega^k z^{-\frac{1}{3}})^{-\beta-3} e^{-\frac{3}{2} \omega^k z^{2/3} - \tau \omega^{2k} z^{1/3}} \left(1 + \mathcal{O}\left(z^{-\frac{1}{3}}\right)\right). \end{aligned}$$

The final step in the proof is the identification of the steepest descent paths, and the deformation of the Γ_k into steepest descent paths. This gives us expressions for the q_k in each sector. For the second and third row of Ψ we remark that by (2.13) differentiation of the q_k is equivalent to increasing β by 1.

The final expansion for Ψ then turns out to be exactly as in (2.7). \square

For the further analysis we also need to know the next order term in the expansion (2.7).

Lemma 2.3. *We have as $z \rightarrow \infty$ with $\pm \operatorname{Im} z > 0$*

$$\begin{aligned} \Psi(z) = \sqrt{\frac{2\pi}{3}} e^{\frac{\tau^2}{6}} z^{\frac{\beta}{3}} \begin{pmatrix} z^{\frac{1}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-\frac{1}{3}} \end{pmatrix} \Omega_{\pm} \\ \times \left(I + (\Psi_1)_{\pm} z^{-\frac{1}{3}} + \mathcal{O}\left(z^{-\frac{2}{3}}\right) \right) B_{\pm} e^{\Theta(z)}, \end{aligned} \quad (2.22)$$

where the constant matrices $(\Psi_1)_{\pm}$ are given by

$$(\Psi_1)_+ = -\frac{\tau}{3} \left(\frac{\tau^2}{9} + \beta + 1 \right) \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} - \frac{\tau}{9} \begin{pmatrix} 0 & \omega^2 - \omega & 1 - \omega \\ \omega^2 - 1 & 0 & 1 - \omega \\ 1 - \omega^2 & \omega^2 - \omega & 0 \end{pmatrix} \quad (2.23)$$

$$(\Psi_1)_- = -\frac{\tau}{3} \left(\frac{\tau^2}{9} + \beta + 1 \right) \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} - \frac{\tau}{9} \begin{pmatrix} 0 & \omega^2 - \omega & \omega^2 - 1 \\ 1 - \omega & 0 & 1 - \omega^2 \\ \omega - 1 & \omega - \omega^2 & 0 \end{pmatrix} \quad (2.24)$$

Proof. The first row of Ψ_1 can be found in a straightforward way by expressing that the entries in the first row of $\Psi(z)$ solve the differential equation (2.11). For the second and third rows we increase β in (2.11) by 1 and 2, respectively. \square

3. Second tool: modified equilibrium problem and Riemann surface

3.1. Modified equilibrium problem

One of the transformations in the Deift/Zhou steepest descent analysis of the RH problem for (multiple) orthogonal polynomials is typically based on the limiting zero distribution of the associated polynomials. In our situation these are given by the vector equilibrium problem from Definition 1.3. As explained in Subsection 1.3, the endpoint $b = b_a$ of one of the supports is varying with a and tends to 0 as $a \rightarrow -1$. Working with measures with varying supports around 0 would cause major technical problems. Therefore, following [10, 12], we use a modified equilibrium problem, where the positivity of the measures is not required. So we will be dealing with signed measures.

Definition 3.1. The modified equilibrium problem asks for two signed measures μ_1 and μ_2 minimizing the energy (1.6) among all signed measures with $\operatorname{supp}(\mu_1) \subset [a, 0]$, $\operatorname{supp}(\mu_2) \subset [0, 1]$ and $\int d\mu_1 = \frac{1}{2}$, $\int d\mu_2 = \frac{1}{2}$.

The modified equilibrium measures μ_1 and μ_2 are unique, and have full supports $[a, 0]$ and $[0, 1]$, respectively. In fact, the modification of the equilibrium

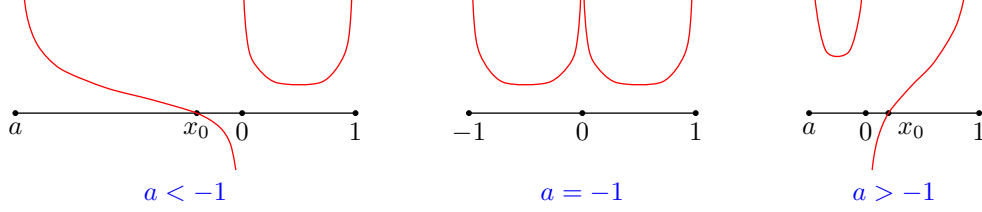


FIGURE 5. A sketch of the modified equilibrium densities in the three cases $a < -1$, $a = -1$ and $a > -1$. The modified equilibrium density becomes negative near 0 on the larger of the two intervals.

problem comes down to forcing the equilibrium measures to have full supports, at the expense of losing positivity of one of the measures near 0.

In the symmetric case $a = -1$ the modified equilibrium measures coincide with the usual equilibrium measures. In this case the equilibrium densities ψ_1 and ψ_2 are positive on the whole supports, and around 0 they blow up like an inverse cube root.

In the general case $a \neq -1$ the equilibrium density on the smaller interval is positive on the full interval. The density on the larger interval becomes negative in an interval between 0 and x_0 where x_0 depends on a in such a way that

$$x_0 = x_0(a) = \frac{(a+1)^3}{108} + \mathcal{O}(a+1)^4 \quad \text{as } a \rightarrow -1, \quad (3.1)$$

see (3.11) below. For $a \neq -1$ both densities $\psi_1(x)$ and $\psi_2(x)$ behave like $x^{-2/3}$ as $x \rightarrow 0$. A sketch of the densities is given in Figure 5.

Define the logarithmic potentials U^μ of a (signed) measure μ by

$$U^\mu(x) = \int \log \frac{1}{|x-y|} d\mu(y), \quad x \in \mathbb{C}. \quad (3.2)$$

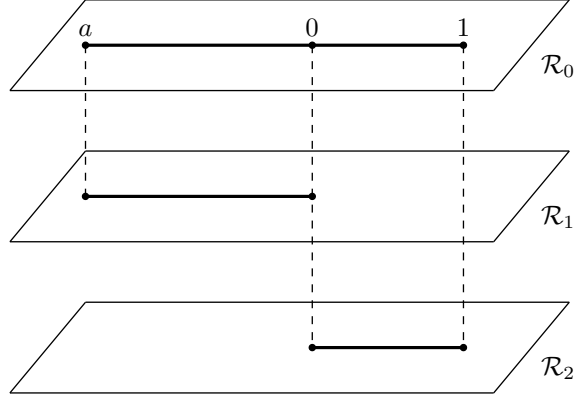
The Euler-Lagrange variational conditions [14, 33] for μ_1 and μ_2 then say that there exist constants $l_1, l_2 \in \mathbb{R}$ such that

$$\begin{aligned} 2U^{\mu_1}(x) + U^{\mu_2}(x) &= l_1, \quad \text{for } x \in [a, 0], \\ U^{\mu_1}(x) + 2U^{\mu_2}(x) &= l_2, \quad \text{for } x \in [0, 1], \end{aligned} \quad (3.3)$$

and these conditions characterize the modified equilibrium measures. For the non-modified equilibrium measures we would have an inequality instead of equality for x in the gap of the supports. The fact that for μ_1 and μ_2 the Euler-Lagrange variational conditions have such a simple form on the full intervals $[a, 0]$ and $[0, 1]$ will be important for the further analysis.

3.2. Riemann surface

The modified equilibrium problem is easiest to analyze by means of an appropriate three-sheeted Riemann surface \mathcal{R} . Define \mathcal{R} by taking three copies of the Riemann

FIGURE 6. The sheets of the Riemann surface \mathcal{R}

sphere $\overline{\mathbb{C}}$ with cuts

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus [a, 1], \quad \mathcal{R}_1 := \overline{\mathbb{C}} \setminus [a, 0], \quad \mathcal{R}_2 := \overline{\mathbb{C}} \setminus [0, 1], \quad (3.4)$$

and gluing them together along these cuts in the usual crosswise manner, see Figure 6.

The Riemann surface has genus zero and can be defined by the polynomial equation

$$4a\xi^3 - 2(a+1)z\xi^3 - 3(a-1)z\xi^2 + (a-1)z = 0. \quad (3.5)$$

Solving for z , we find a rational function

$$z = z(\xi) = \frac{4a\xi^3}{2(a+1)\xi^3 + 3(a-1)\xi^2 - (a-1)}, \quad (3.6)$$

which defines a conformal map from $\xi \in \overline{\mathbb{C}}$ to $z \in \mathcal{R}$, so that the branch points a , 0 , and 1 of \mathcal{R} correspond to $\xi = -1, 0, 1$, respectively. The restriction of the inverse mapping of (3.6) to the sheet \mathcal{R}_i is denoted by ξ_i

$$\xi_i : \mathcal{R}_i \rightarrow \mathbb{C}.$$

The ξ -functions map the sheets of \mathcal{R} to certain domains

$$\tilde{\mathcal{R}}_i := \xi_i(\mathcal{R}_i)$$

of $\overline{\mathbb{C}}$. Then $\tilde{\mathcal{R}}_1$ and $\tilde{\mathcal{R}}_2$ are bounded, while $\tilde{\mathcal{R}}_0$ is unbounded. We use γ_1^\pm and γ_2^\pm to denote the arcs bounding $\tilde{\mathcal{R}}_1$ and $\tilde{\mathcal{R}}_2$ with clockwise orientation as in Figure 8 below.

3.3. Properties of the modified equilibrium problem

We can make use of the Riemann surface to prove the following properties of the modified equilibrium measures. We will not give all details in the following calculations.

The modified equilibrium measures are described in terms of the algebraic equation

$$\zeta^3 - \frac{3z - 2z^*(a) - 1 - a}{4z(z-a)(z-1)}\zeta - \frac{z - z^*(a)}{4z^2(z-a)(z-1)} = 0 \quad (3.7)$$

where $z^* = z^*(a)$ is a certain solution of

$$64(z^*)^3 - 48(a+1)(z^*)^2 - (15a^2 - 78a + 15)z^* - (a+1)^3 = 0. \quad (3.8)$$

It can be shown that (3.8) has three distinct real solutions if $a < 0$. We use $z^*(a)$ to denote the middle one of the three solutions and this is the value that is used in (3.7).

Proposition 3.2. *For $a < 0$, the following hold.*

(a) *The three solutions of (3.7) are given by*

$$\begin{aligned} \zeta_0(z) &= \int_a^0 \frac{\psi_1(x)}{z-x} dx + \int_0^1 \frac{\psi_2(x)}{z-x} dx, \\ \zeta_1(z) &= - \int_a^0 \frac{\psi_1(x)}{z-x} dx, \\ \zeta_2(z) &= - \int_0^1 \frac{\psi_2(x)}{z-x} dx, \end{aligned} \quad (3.9)$$

where ψ_1 and ψ_2 are the densities of the modified equilibrium measures.

(b) *The densities satisfy*

$$\begin{aligned} \psi_1(x) &= \frac{1}{2\pi i} (\zeta_{1,+}(x) - \zeta_{1,-}(x)), \quad x \in (a, 0), \\ \psi_2(x) &= \frac{1}{2\pi i} (\zeta_{2,+}(x) - \zeta_{2,-}(x)), \quad x \in (0, 1). \end{aligned} \quad (3.10)$$

(c) *There is $x_0 = x_0(a) \in (a, 1)$ with the same sign as $a+1$ such that*

- *if $-1 < a < 0$, then $\psi_1(x) > 0$ for $x \in (a, 0)$ and $\psi_2(x) < 0$ if and only if $0 < x < x_0$,*
- *if $a < -1$, then $\psi_2(x) > 0$ for $x \in (0, 1)$ and $\psi_1(x) < 0$ if and only if $x_0 < x < 0$.*

See also Figure 5.

(d) *We have*

$$x_0(a) = \frac{(a+1)^3}{108} + \mathcal{O}(a+1)^4 \quad \text{as } a \rightarrow -1. \quad (3.11)$$

Proof. Let ζ_j , $j = 0, 1, 2$ be defined by (3.9), so that we clearly have

$$\zeta_0(z) + \zeta_1(z) + \zeta_2(z) = 0 \quad (3.12)$$

It follows from the variational conditions (3.3) that $\zeta_{0,+} = \zeta_{1,-}$ on $(a, 0)$ and $\zeta_{0,-} = \zeta_{2,-}$ on $(0, 1)$.

Thus if we consider ζ_j as a function defined on the sheet \mathcal{R}_j for $j = 0, 1, 2$, then this function extends to a meromorphic function on \mathcal{R} . Since (due to the normalization $\int d\mu_1 = \int d\mu_2 = 1/2$)

$$\zeta_0(z) = z^{-1} + \mathcal{O}(z^{-2}), \quad \zeta_j(z) = -\frac{1}{2}z^{-1} + \mathcal{O}(z^{-2}), \quad (3.13)$$

as $z \rightarrow \infty$, the meromorphic function has simple zeros at the three points at infinity. There are simple poles at a and 1 and a possible double pole at 0. In addition there is a fourth simple zero at a point z^* .

Then the product $\zeta_0\zeta_1\zeta_2$ is a rational function in the complex plane with a zero at z^* , simple poles at a , -1 , a double pole at 0, and it behaves as $\frac{1}{4}z^{-3}$ as $z \rightarrow \infty$. This means that

$$\zeta_0(z)\zeta_1(z)\zeta_2(z) = \frac{z - z^*}{4z^2(z - a)(z - 1)}. \quad (3.14)$$

Similar considerations show that

$$\zeta_0(z)\zeta_1(z) + \zeta_0(z)\zeta_2(z) + \zeta_1(z)\zeta_2(z) = -\frac{3z - q}{4z(z - a)(z - 1)} \quad (3.15)$$

for some q . Thus ζ_j , $j = 1, 2, 3$ are the three solutions of the algebraic equation

$$\zeta^3 - \frac{3z - q}{4z(z - a)(z - 1)}\zeta - \frac{z - z^*}{4z^2(z - a)(z - 1)} = 0 \quad (3.16)$$

Inserting $\zeta = \zeta_1(z) = -1/(2z) + \mathcal{O}(z^{-2})$ into (3.16) shows that $q = 2z^* + a + 1$, which gives us the equation (3.7).

The discriminant of (3.7) with respect to ζ has the form

$$\frac{Q_2(z)}{16z^4(z - a)^3(z - 1)^3}$$

where $Q_2(z)$ is a certain quadratic polynomial in z that we calculated with Maple. The poles a , 0 and 1 of the discriminant correspond to the branch points of the Riemann surface. The quadratic polynomial should have a double zero, since otherwise there would be more branch points. This leads to a condition on z^* , which turns out to be given by (3.8). Again we made these calculations with Maple. This proves part (a) of the proposition.

The relevant solution z^* of (3.8) is the one that is 0 for $a = -1$. This solution is then well-defined as a real analytic function for $a \in (-\infty, 0)$. We have

$$z^*(a) = -\frac{(a + 1)^3}{108} + \mathcal{O}((a + 1)^4) \quad \text{as } a \rightarrow -1, \quad (3.17)$$

which can be obtained from (3.8). The double root of $Q_2(z)$ turns out to be equal to

$$x_0(a) = \frac{(1 + a)^3 + (6a^2 - 42a + 6)z^*(a) - 15(1 + a)(z^*(a))^2 + 8(z^*(a))^3}{18(1 - a + a^2 - 2(1 + a)z^*(a) + (z^*(a))^2)},$$

which can be shown to also satisfy a cubic equation

$$(27a^2 - 46a + 27)x_0^3 - 3(a+1)(9a^2 - 14a + 9)x_0^2 + 3a(11a^2 - 14a + 11)x_0 - a(a+1)^3 = 0. \quad (3.18)$$

There are three real distinct solutions of (3.18) if $a < 0$ and $x_0(a)$ is the middle one. The expansion (3.11) follows from (3.18) and part (d) follows.

Part (b) follows immediately from part (a) and the Sokhotskii-Plemelj formulas that tell us how to recover the density of a measure from its Cauchy transform.

Finally, to prove part (c), we suppose that $-1 < a < 0$. It can then be shown from the above formulas (it is not immediate, however) that $0 < x_0(a) < 1$. Since $x_0(a)$ is a zero of the discriminant, the cubic equation (3.7) has a double solution if $z = x_0(a)$. Since $0 < x_0(a) < 1$, we have $\xi_{0,+}(x_0(a)) = \xi_{2,+}(x_0(a))$ and $\xi_1(x_0(a))$ is real. Thus we have $\xi_{0,+}(x_0(a)) = \xi_{2,+}(x_0(a))$, which means since $\xi_{0,+} = \xi_{2,-}$ on $(0, 1)$, that ψ_2 vanishes at $x_0(a)$ by (3.10). Since $x_0(a)$ is the only zero of the discriminant in $(a, 1)$, it also follows that $x_0(a)$ is the only zero of ψ_2 , and that ψ_1 has no zeros. Thus $\psi_1 > 0$ on $(a, 0)$. It is a consequence of the fact that the point 0 of the Riemann surface is a double pole, that ψ_1 and ψ_2 have opposite signs near 0. Thus $\psi_2(x) < 0$ for $0 < x < x_0(a)$ and part (c) of the proposition is proved in case $-1 < a < 0$.

The proof for $a < -1$ is similar. \square

4. Steepest descent analysis of the RH problem

4.1. First transformation

We start from the RH problem for Y with $n_1 = n_2 = n$. We also take $a < 0$ close to -1 but for the moment it is arbitrary and fixed. We use the modified equilibrium measures μ_1 and μ_2 that are supported on the two intervals $[a, 0]$ and $[0, 1]$ respectively.

Define g -functions by

$$g_j(z) = \int \log(z - s) d\mu_j(s), \quad j = 1, 2, \quad (4.1)$$

where we use the main branch of the logarithm. Hence g_1 is defined with a branch cut on $(-\infty, 0]$ and g_2 with a branch cut on $(-\infty, 1]$. The boundary values of the g -functions along the real axis are given by

$$\begin{aligned} g_{1,\pm}(x) &= -U^{\mu_1}(x) \pm \pi i \int_x^0 d\mu_1(s), \\ g_{2,\pm}(x) &= -U^{\mu_2}(x) \pm \pi i \int_x^1 d\mu_2(s), \end{aligned} \quad (4.2)$$

were U^{μ_1} and U^{μ_2} are the logarithmic potentials (3.2). From (3.3) and (4.2) we obtain

$$\begin{aligned} g_{1,+}(x) + g_{1,-}(x) + g_{2,\pm}(x) &= -l_1 \pm \frac{1}{2}\pi i, \quad x \in [a, 0], \\ g_1(x) + g_{2,+}(x) + g_{2,-}(x) &= -l_2, \quad x \in [0, 1], \end{aligned} \quad (4.3)$$

Now define the first transformation $Y \mapsto T$ as

$$\begin{aligned} T(z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2n(l_1 + \frac{1}{2}\pi i)} & 0 \\ 0 & 0 & e^{-2nl_2} \end{pmatrix} Y(z) \\ &\quad \times \begin{pmatrix} e^{-2n(g_1(z) + g_2(z))} & 0 & 0 \\ 0 & e^{2n(g_1(z) + l_1 + \frac{1}{2}\pi i)} & 0 \\ 0 & 0 & e^{2n(g_2(z) + l_2)} \end{pmatrix}. \end{aligned} \quad (4.4)$$

This transformation normalizes the RH problem at ∞ , since the g -functions behave like $\frac{1}{2}\log z + O(z^{-1})$ as $z \rightarrow \infty$. Thus $T(z) = I + O(z^{-1})$ as $z \rightarrow \infty$. The jumps for T are conveniently expressed in terms of the two functions φ_j , $j = 1, 2$ defined by

$$\begin{aligned} \varphi_1(z) &:= -2g_1(z) - g_2(z) - l_1 + \begin{cases} \frac{1}{2}\pi i & \text{for } \operatorname{Im} z > 0, \\ -\frac{1}{2}\pi i & \text{for } \operatorname{Im} z < 0, \end{cases} \\ \varphi_2(z) &:= -2g_2(z) - g_1(z) - l_2 + \begin{cases} \pi i & \text{for } \operatorname{Im} z > 0, \\ -\pi i & \text{for } \operatorname{Im} z < 0. \end{cases} \end{aligned} \quad (4.5)$$

Then by (2.1), (4.4), and (4.3) one obtains the jump matrices

$$J_T(x) = \begin{pmatrix} e^{2n\varphi_{1,+}(x)} & w_1(x) & 0 \\ 0 & e^{2n\varphi_{1,-}(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } x \in (a, 0), \quad (4.6)$$

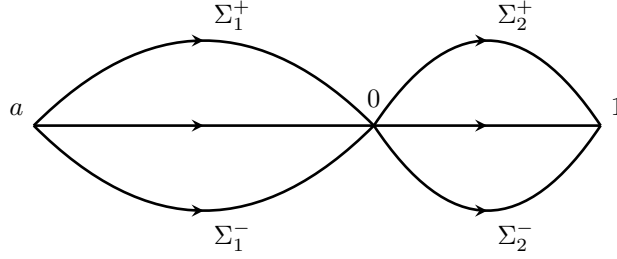
$$J_T(x) = \begin{pmatrix} e^{2n\varphi_{2,+}(x)} & 0 & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & e^{2n\varphi_{2,-}(x)} \end{pmatrix} \quad \text{for } x \in (0, 1). \quad (4.7)$$

Thus T satisfies the following RH problem.

- T is an analytic 3×3 matrix valued function on $\mathbb{C} \setminus [a, 1]$,
- T satisfies jump conditions $T_+(x) = T_-(x)J_T(x)$ for $x \in (a, 0) \cup (0, 1)$, with J_T given by (4.6) and (4.7),
- T is normalized at infinity:

$$T(z) = I + O(z^{-1}) \quad \text{as } z \rightarrow \infty, \quad (4.8)$$

- near a , 0 and 1 the function T has the same behavior as Y , see (2.3), (2.4), and (2.5).

FIGURE 7. The contour Σ_S and the lenses around $[a, 0]$ and $[0, 1]$.

4.2. Second transformation: opening of the lenses

The functions $e^{2n\varphi_{k,\pm}}$, $k = 1, 2$ appearing in the jump matrices J_T in (4.6) and (4.7) are rapidly oscillating for large n , since the boundary values $\varphi_{k,\pm}$ are purely imaginary. One may easily check that

$$\varphi_{1,\pm}(x) = \mp 2\pi i \int_x^0 d\mu_1(s) \quad \text{for } x \in (a, 0), \quad (4.9)$$

$$\varphi_{2,\pm}(x) = \pm 2\pi i \int_0^x d\mu_2(s) \quad \text{for } x \in (0, 1). \quad (4.10)$$

The oscillations are turned into exponential decay by the so-called opening of the lenses. Choose smooth paths Σ_1^\pm connecting a and 0 with Σ_1^+ in the upper half-plane and Σ_1^- in the lower half plane. Similarly, choose paths Σ_2^+ and Σ_2^- connecting 0 and 1 . Define

$$\Sigma_S := \Sigma_1^\pm \cup \Sigma_2^\pm \cup [a, 1]. \quad (4.11)$$

The intervals $[a, 0]$, $[0, 1]$ and the paths Σ_i^\pm define 4 bounded regions that are referred to as the lenses around $[a, 0]$ and $[0, 1]$, see Figure 7.

For $j = 1, 2$, let V_j be a simply connected neighborhood of Δ_j such that the analytic factor h_j in the weight function w_j is analytic and non-zero in V_j . We assume that $\Sigma_j^\pm \subset V_j$ for $j = 1, 2$. Then w_j has an analytic continuation from Δ_j to V_j with some cuts, which we also denote by w_j :

$$\begin{aligned} w_1(z) &= (z-a)^\alpha (-z)^\beta h_1(z), & \text{for } z \in V_1 \setminus ((-\infty, a] \cup [0, \infty)), \\ w_2(z) &= z^\beta (1-z)^\gamma h_2(z), & \text{for } z \in V_2 \setminus ((-\infty, 0] \cup [1, \infty)). \end{aligned} \quad (4.12)$$

Then, following [29, section 4], we define the next transformation $T \mapsto S$ by

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 & 0 \\ -w_1(z)^{-1}e^{2n\varphi_1(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{array}{l} z \text{ in the upper part of} \\ \text{the lens around } [a, 0], \end{array} \\ T(z) \begin{pmatrix} 1 & 0 & 0 \\ w_1(z)^{-1}e^{2n\varphi_1(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{array}{l} z \text{ in the lower part of} \\ \text{the lens around } [a, 0], \end{array} \end{cases} \quad (4.13)$$

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -w_2(z)^{-1}e^{2n\varphi_2(z)} & 0 & 1 \end{pmatrix}, & \begin{array}{l} z \text{ in the upper part of} \\ \text{the lens around } [0, 1], \end{array} \\ T(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ w_2(z)^{-1}e^{2n\varphi_2(z)} & 0 & 1 \end{pmatrix}, & \begin{array}{l} z \text{ in the lower part of} \\ \text{the lens around } [0, 1], \end{array} \end{cases} \quad (4.14)$$

and

$$S(z) = T(z) \quad \text{elsewhere.} \quad (4.15)$$

It is clear that this transformation does not affect the behavior at infinity. The jump matrix J_S for S on the intervals $(a, 0)$ and $(0, 1)$ are

$$J_S(x) = \begin{cases} \begin{pmatrix} 0 & w_1(x) & 0 \\ -w_1(x)^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & x \in (a, 0), \\ \begin{pmatrix} 0 & 0 & w_2(x) \\ 0 & 1 & 0 \\ -w_2(x)^{-1} & 0 & 0 \end{pmatrix}, & x \in (0, 1). \end{cases} \quad (4.16)$$

The transformation has introduced jumps on Σ_j^\pm which are

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ w_1(z)^{-1}e^{2n\varphi_1(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_1^\pm, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ w_2(z)^{-1}e^{2n\varphi_2(z)} & 0 & 1 \end{pmatrix}, & z \in \Sigma_2^\pm. \end{cases} \quad (4.17)$$

Finally, the behavior near a , 0 and 1 changes because of the factors w_j^{-1} in the transformation (4.13)-(4.14). The Riemann-Hilbert problem for S then reads:

- S is analytic on $\mathbb{C} \setminus \Sigma_S$,
- S has jumps $S_+ = S_- J_S$, where J_S is given by (4.16) and (4.17),
- $S(z) = I + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$,

- near the endpoints of the intervals S behaves as

$$S(z) = \mathcal{O} \begin{pmatrix} \epsilon_1(z) & \epsilon_2(z) & 1 \\ \epsilon_1(z) & \epsilon_2(z) & 1 \\ \epsilon_1(z) & \epsilon_2(z) & 1 \end{pmatrix}, \quad \text{as } z \rightarrow a, \quad (4.18)$$

where

$$\epsilon_1(z), \epsilon_2(z) = \begin{cases} 1, |z-a|^\alpha & \text{if } \alpha < 0, \\ \log |z-a|, \log |z-a| & \text{if } \alpha = 0, \\ |z-a|^{-\alpha}, 1 & \text{if } \alpha > 0, z \text{ inside the lens,} \\ 1, 1 & \text{if } \alpha > 0, z \text{ outside the lens,} \end{cases}$$

$$S(z) = \mathcal{O} \begin{pmatrix} \epsilon_1(z) & 1 & \epsilon_2(z) \\ \epsilon_1(z) & 1 & \epsilon_2(z) \\ \epsilon_1(z) & 1 & \epsilon_2(z) \end{pmatrix}, \quad \text{as } z \rightarrow 1, \quad (4.19)$$

where

$$\epsilon_1(z), \epsilon_2(z) = \begin{cases} 1, |z-1|^\gamma & \text{if } \gamma < 0, \\ \log |z-1|, \log |z-1| & \text{if } \gamma = 0, \\ |z-1|^{-\gamma}, 1 & \text{if } \gamma > 0, z \text{ inside the lens,} \\ 1, 1 & \text{if } \gamma > 0, z \text{ outside the lens,} \end{cases}$$

$$S(z) = \mathcal{O} \begin{pmatrix} \epsilon_1(z) & \epsilon_2(z) & \epsilon_2(z) \\ \epsilon_1(z) & \epsilon_2(z) & \epsilon_2(z) \\ \epsilon_1(z) & \epsilon_2(z) & \epsilon_2(z) \end{pmatrix}, \quad \text{as } z \rightarrow 0, \quad (4.20)$$

where

$$\epsilon_1(z), \epsilon_2(z) = \begin{cases} 1, |z|^\beta & \text{if } \beta < 0, \\ \log |z|, \log |z| & \text{if } \beta = 0, \\ |z|^{-\beta}, 1 & \text{if } \beta > 0, z \text{ inside the lenses,} \\ 1, 1 & \text{if } \beta > 0, z \text{ outside the lenses.} \end{cases}$$

For later analysis it will be important to know how $\operatorname{Re} \varphi_1$ and $\operatorname{Re} \varphi_2$ behave on the lips of the lenses. From (4.17) we see that we would like to have

$$\operatorname{Re} \varphi_j(z) < 0 \quad \text{for } z \in \Sigma_j^\pm, \quad (4.21)$$

for $j = 1, 2$. The inequality (4.21) will indeed hold if μ_j is a positive measure, and this can be proven using the Cauchy-Riemann equations. Recall however, that μ_1 and μ_2 are signed measures. The inequality (4.21) will be violated for z on the parts of Σ_j that are close to the interval where μ_j is negative. By Proposition (3.2) we have that μ_1 is negative on $(x_0(a), 0)$ if $a < -1$, and that μ_2 is negative near $(0, x_0(a))$ if $-1 < a < 0$, where $x_0(a) = \mathcal{O}((a+1)^3)$ as $a \rightarrow -1$.

We write $\varphi_j(z; a)$ to emphasize the dependence on a .

Lemma 4.1. *There exist positive constants C_0 and C_1 , independent of a , such that for every a sufficiently close to -1 , we have*

$$\operatorname{Re} \varphi_j(z; a) \leq C_0|a+1||z|^{1/3} - C_1|z|^{2/3}, \quad z \in \Sigma_j^\pm, |z| < 1/2. \quad (4.22)$$

for $j = 1, 2$,

Proof. We have by the definitions (3.9), (4.1), and (4.5) that $\varphi'_j = \zeta_j - \zeta_0$ for $j = 1, 2$. The constant in (4.5) is taken so that $\varphi_j(0)$ which means that

$$\varphi_j(z; a) = \int_0^z (\zeta_j(s; a) - \zeta_0(s; a)) ds \quad (4.23)$$

where $\zeta_0, \zeta_1, \zeta_2$ are the three solution of the cubic equation (3.7), where we emphasize the dependence on a .

As $s \rightarrow 0$ with $\operatorname{Im} s > 0$ we can compute from (3.7) that

$$\zeta_0(s; a) = c_0 \omega^2 s^{-2/3} + c_1 \omega s^{-1/3} + \mathcal{O}(1), \quad (4.24)$$

$$\zeta_1(s; a) = c_0 s^{-2/3} + c_1 s^{-1/3} + \mathcal{O}(1), \quad (4.25)$$

$$\zeta_2(s; a) = c_0 \omega s^{-2/3} + c_1 \omega^2 s^{-1/3} + \mathcal{O}(1) \quad (4.26)$$

uniformly for a close to -1 , with real constants $c_0 = (-z^*(a)/(4a))^{1/3}$ and $c_1 = -(2z^*(a) + a + 1)/(12ac_0)$. Because of the behavior (3.17) of $z^*(a)$ we have that

$$\begin{aligned} c_0 &= c_0(a) = -\frac{2^{2/3}}{12}(a+1) + \mathcal{O}((a+1)^2) \\ c_1 &= c_1(a) = -\frac{2^{1/3}}{2} + \mathcal{O}(a+1) \end{aligned} \quad (4.27)$$

as $a \rightarrow -1$.

Using (4.24) in (4.23) we find

$$\varphi_2(z; a) = 3^{3/2} i c_0(a) z^{1/3} - \frac{1}{2} 3^{3/2} i c_1(a) z^{2/3} + \mathcal{O}(z) \quad (4.28)$$

as $z \rightarrow 0$ with $\operatorname{Im} z > 0$. By (4.27) we have that $c_1(a)$ tends to a negative constant as $a \rightarrow -1$. Since we may assume that the lens is opened with a positive angle at 0, we find that

$$\operatorname{Re} \left(-\frac{3}{2} \sqrt{3} i c_1(a) z^{2/3} \right) \leq -C_1 |z|^{2/3}, \quad z \in \Sigma_2^+$$

for some constant $C_1 > 0$ independent of a . Using this in (4.28) we obtain (4.23) for $j = 2$ and $z \in \Sigma_2^+$ in a fixed size neighborhood of $z = 0$, say $|z| < r_0$. The inequality (4.23) then also holds for $|z| < 1/2$ (maybe with different constant C_1), since $\varphi_2(z; a) \rightarrow \varphi_2(z; -1)$ as $a \rightarrow -1$ uniformly for $r_0 \leq |z| \leq 1/2$, and $\operatorname{Re} \varphi_2(z; -1) < -C_3 < 0$ for $z \in \Sigma_2^+$, $r_0 \leq |z| \leq 1/2$, and some $C_3 > 0$.

The inequality (4.23) for $j = 2$ and $z \in \Sigma_2^-$ and for $j = 1$ follow in a similar way. \square

It follows from (4.22) that we indeed have that $\operatorname{Re} \varphi_j(z; a) < 0$ for $z \in \Sigma_j^\pm$, except for z in a small exceptional neighborhood of $z = 0$, whose radius shrinks as $\mathcal{O}((a+1)^3)$ as $a \rightarrow -1$.

4.3. Outer parametrix

The next step is to construct an approximation to S . This so-called parametrix consists of an outer parametrix N that gives an approximation away from the endpoints a , 0 and 1 and local parametrices P around each of the endpoints.

4.3.1. Riemann-Hilbert problem for N . The outer parametrix N should satisfy

- N is analytic on $\mathbb{C} \setminus [a, 1]$,
- N satisfies the jump conditions

$$N_+(x) = \begin{cases} N_-(x) \begin{pmatrix} 0 & w_1(x) & 0 \\ -w_1(x)^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } x \in (a, 0), \\ N_-(x) \begin{pmatrix} 0 & 0 & w_2(x) \\ 0 & 1 & 0 \\ -w_2(x)^{-1} & 0 & 0 \end{pmatrix} & \text{if } x \in (0, 1), \end{cases} \quad (4.29)$$

- N is normalized at infinity:

$$N(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \rightarrow \infty. \quad (4.30)$$

4.3.2. Solution in a special case. First we will find a solution \tilde{N} to this problem for the case that both weights w_1 and w_2 are identically 1 on their respective intervals. We solve the problem for \tilde{N} by using the Riemann surface \mathcal{R} introduced before. There is a similar construction in [9] and so we do not go into much detail here.

The function $\xi = \xi_j(z)$ maps the sheet \mathcal{R}_j of the Riemann surface onto the domain $\tilde{\mathcal{R}}_j$ as shown in Figure 8 that are separated by two closed contours γ_1 and γ_2 that we orient in the clockwise direction. We write

$$p_j = p_j(a) = \xi_j(\infty), \quad j = 0, 1, 2.$$

A solution \tilde{N} is given in the form

$$\tilde{N}(z) = \begin{pmatrix} \tilde{N}_0(\xi_0(z)) & \tilde{N}_0(\xi_1(z)) & \tilde{N}_0(\xi_2(z)) \\ \tilde{N}_1(\xi_0(z)) & \tilde{N}_1(\xi_1(z)) & \tilde{N}_1(\xi_2(z)) \\ \tilde{N}_2(\xi_0(z)) & \tilde{N}_2(\xi_1(z)) & \tilde{N}_2(\xi_2(z)) \end{pmatrix}, \quad (4.31)$$

with the following functions \tilde{N}_j that are analytic on $\overline{\mathbb{C}} \setminus (\gamma_1^\pm \cup \gamma_2^\pm)$

$$\tilde{N}_j(\xi) = \frac{p_j(p_j^2 - 1)^{1/2}}{\prod_{\substack{i=0 \\ i \neq j}} (p_j - p_i)} \frac{1}{\xi(\xi^2 - 1)^{1/2}} \prod_{\substack{i=0 \\ i \neq j}}^2 (\xi - p_i) \quad (4.32)$$

with appropriate modifications if $p_0 = \infty$ (which happens if $a = -1$). The branch cut of the square roots $(\xi^2 - 1)^{1/2}$ and $(p_j^2 - 1)^{1/2}$ in (4.32) is defined along $\gamma_1^+ \cup \gamma_2^+$, see Figure 8. It can then be checked that the matrix function \tilde{N} defined by (4.31)–(4.32) indeed satisfies the conditions in the Riemann-Hilbert problem for \tilde{N} .

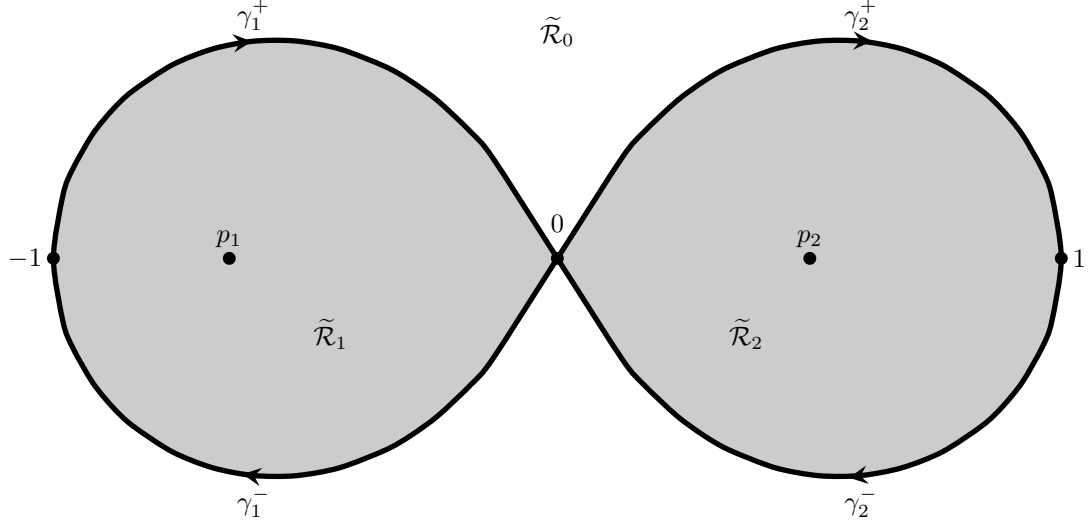


FIGURE 8. The images of the sheets of \mathcal{R} under the functions ξ_j . The location of $p_1 = \xi_1(\infty)$ and $p_2 = \xi_2(\infty)$ depend on the choice of a , but the contours γ_1^\pm and γ_2^\pm do not.

4.3.3. Solution in general case. Now we turn to the problem for N , with jumps involving w_1 and w_2 . This can be solved using analogues to the Szegő function as in [27, 29]. We look for three functions D_0 , D_1 and D_2 satisfying

- (a) D_0 is analytic and non-zero in $\overline{\mathbb{C}} \setminus [a, 1]$,
- (b) D_1 is analytic and non-zero in $\overline{\mathbb{C}} \setminus [a, 0]$,
- (c) D_2 is analytic and non-zero in $\overline{\mathbb{C}} \setminus [0, 1]$,
- (d) D_0 , D_1 and D_2 have limiting values on $(a, 0)$ and $(0, 1)$ such that

$$\frac{D_{1,+}}{D_{0,-}} = \frac{D_{1,-}}{D_{0,+}} = w_1 \quad \text{on } (a, 0)$$

and

$$\frac{D_{2,+}}{D_{0,-}} = \frac{D_{2,-}}{D_{0,+}} = w_2 \quad \text{on } (0, 1).$$

Having D_0 , D_1 and D_2 we define N by

$$N(z) := \begin{pmatrix} D_0(\infty)^{-1} & 0 & 0 \\ 0 & D_1(\infty)^{-1} & 0 \\ 0 & 0 & D_2(\infty)^{-1} \end{pmatrix} \tilde{N}(z) \begin{pmatrix} D_0(z) & 0 & 0 \\ 0 & D_1(z) & 0 \\ 0 & 0 & D_2(z) \end{pmatrix}. \quad (4.33)$$

We will abbreviate this as

$$N(z) = D_\infty^{-1} \tilde{N}(z) D(z), \quad (4.34)$$

and one can check that N indeed solves the Riemann-Hilbert problem for N .

From the jump properties of D_0 , D_1 and D_2 it follows that

$$(D_0 D_1 D_2)_+ = (D_0 D_1 D_2)_- \quad \text{on } (a, 0) \text{ and } (0, 1).$$

Thus $D_0 D_1 D_2$ is analytic across these cuts. We also make sure that the possible singularities at a , 0 and 1 are removable. Then $D_0 D_1 D_2$ is a constant and we can choose a normalization such that

$$D_0 D_1 D_2 \equiv 1.$$

4.3.4. Szegő functions. In order to find D_0 , D_1 and D_2 we write

$$D_j(z) = \mathcal{D}(\xi_j(z)), \quad j = 0, 1, 2, \quad (4.35)$$

for some yet to be determined function \mathcal{D} on the ξ -Riemann sphere. Recall that ξ_0 , ξ_1 and ξ_2 are the mapping functions from the respective sheets of the Riemann surface \mathcal{R} to the Riemann sphere.

Then \mathcal{D} has to satisfy

- (a) $\mathcal{D} : \overline{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2) \rightarrow \mathbb{C}$ is analytic and non-zero.
- (b) On $\gamma_1 \cup \gamma_2$ there is a jump

$$\mathcal{D}_+(\xi) = w_j(z) \mathcal{D}_-(\xi), \quad \xi \in \gamma_j, \quad j = 1, 2, \quad (4.36)$$

where $z = z(\xi)$ is related to ξ by (3.6).

Then by taking logarithms we get

$$\log \mathcal{D}_+(\xi) = \log w_j(z) + \log \mathcal{D}_-(\xi), \quad \xi \in \gamma_j^+, \quad j = 1, 2, \quad (4.37)$$

which by the Sokhotskii Plemelj formula is solved by the Cauchy transforms

$$\log \mathcal{D}(\xi) = \frac{1}{2\pi i} \left(\int_{\gamma_1} \frac{\log w_1(z(s))}{s - \xi} ds + \int_{\gamma_2} \frac{\log w_2(z(s))}{s - \xi} ds \right) + C_1$$

where C_1 is an arbitrary constant. Thus

$$\mathcal{D}(\xi) = C \exp \left[\frac{1}{2\pi i} \left(\int_{\gamma_1} \frac{\log w_1(z(s))}{s - \xi} ds + \int_{\gamma_2} \frac{\log w_2(z(s))}{s - \xi} ds \right) \right]$$

with $C = e^{C_1}$ and

$$D_j(z) = C \exp \left[\frac{1}{2\pi i} \left(\int_{\gamma_1} \frac{\log w_1(z(s))}{s - \xi_j(z)} ds + \int_{\gamma_2} \frac{\log w_2(z(s))}{s - \xi_j(z)} ds \right) \right], \quad (4.38)$$

for $j = 0, 1, 2$. The constant C can be taken so that $D_0 D_1 D_2 \equiv 1$. This completes the construction of D_0 , D_1 and D_2 and therefore of N .

Example. In the case where $h_1 \equiv 1$, $h_2 \equiv 1$, we can evaluate the Szegő functions explicitly. Indeed, we find for

$$\begin{aligned} w_1(x) &= w_1^{(\alpha, \beta)}(x) = (x - a)^\alpha (-x)^\beta, \quad x \in (a, 0), \\ w_2(x) &= w_2^{(\beta, \gamma)}(x) = x^\beta (1 - x)^\gamma, \quad x \in (0, 1), \end{aligned} \quad (4.39)$$

that

$$\begin{aligned}
D_0(z) &= C \left(\frac{\xi_0(z) - \xi_1(\infty)}{\xi_0(z) - \xi_1(a)} \right)^\alpha \left(\frac{\xi_0(z) - \xi_2(\infty)}{\xi_0(z) - \xi_2(1)} \right)^\gamma \\
&\quad \times \left(\frac{(\xi_0(z) - \xi_1(\infty))(\xi_0(z) - \xi_2(\infty))}{\xi_0(z)^2} \right)^\beta, \\
D_1(z) &= C \left((z-a) \frac{\xi_1(z) - \xi_1(\infty)}{\xi_1(z) - \xi_1(a)} \right)^\alpha \left(\frac{\xi_1(z) - \xi_2(\infty)}{\xi_1(z) - \xi_2(1)} \right)^\gamma \\
&\quad \times \left((-z) \frac{(\xi_1(z) - \xi_1(\infty))(\xi_1(z) - \xi_2(\infty))}{\xi_1(z)^2} \right)^\beta, \\
D_2(z) &= C \left(\frac{\xi_2(z) - \xi_1(\infty)}{\xi_2(z) - \xi_1(a)} \right)^\alpha \left((1-z) \frac{\xi_2(z) - \xi_2(\infty)}{\xi_2(z) - \xi_2(1)} \right)^\gamma \\
&\quad \times \left(z \frac{(\xi_2(z) - \xi_1(\infty))(\xi_2(z) - \xi_2(\infty))}{\xi_2(z)^2} \right)^\beta.
\end{aligned} \tag{4.40}$$

with appropriate choice of branches for the exponents.

4.3.5. Behavior of Szegő functions near 0. From (3.5) and the choice of branches ξ_0, ξ_1, ξ_2 , we obtain

$$\begin{aligned}
\xi_0(z) &= -C(a)\omega^\mp z^{1/3} + \mathcal{O}(z), \\
\xi_1(z) &= -C(a)z^{1/3} + \mathcal{O}(z), \quad \text{for } \pm \operatorname{Im} z > 0 \\
\xi_2(z) &= -C(a)\omega^\pm z^{1/3} + \mathcal{O}(z),
\end{aligned} \tag{4.41}$$

with

$$C(a) = \left(\frac{a-1}{4a} \right)^{1/3} > 0.$$

We use this in (4.40) together with $\xi_1(a) = -1, \xi_2(1) = 1, \xi_1(\infty) = p_1(a), \xi_2(\infty) = p_2(a)$, to obtain the leading behavior of the Szegő functions at 0 for the case $h_1 \equiv 1, h_2 \equiv 1$. It follows from (4.40) that

$$\begin{aligned}
D_0(z) &= C (-p_1(a))^\alpha p_2(a)^\gamma \left(\frac{-p_1(a)p_2(a)}{C(a)^2} \right)^\beta e^{\pm \beta \pi i / 3} z^{-2\beta/3} (1 + \mathcal{O}(z^{1/3})), \\
D_1(z) &= C (ap_1(a))^\alpha p_2(a)^\gamma \left(\frac{-p_1(a)p_2(a)}{C(a)^2} \right)^\beta z^{\beta/3} (1 + \mathcal{O}(z^{1/3})), \\
D_2(z) &= C (-p_1(a))^\alpha p_2(a)^\gamma \left(\frac{-p_1(a)p_2(a)}{C(a)^2} \right)^\beta e^{\mp \beta \pi i / 3} z^{\beta/3} (1 + \mathcal{O}(z^{1/3}))
\end{aligned} \tag{4.42}$$

as $z \rightarrow 0$ with $\pm \operatorname{Im} z > 0$.

The effect of the analytic factors h_1 and h_2 comes in the form of contour integrals

$$\frac{1}{2\pi i} \oint_{\gamma_j} \frac{\log h_j(z(s))}{s - \xi} ds, \quad j = 1, 2, \quad \xi \in \mathbb{C} \setminus \gamma_j,$$

see (4.38). Because of analyticity we can deform γ_j to a contour γ_j^ε in the region $\tilde{\mathcal{R}}_j$, which leaves the integral unchanged if $\xi \in \mathbb{C} \setminus \tilde{\mathcal{R}}_j$ and picks up a residue contribution of $\log(h_j(z(\xi)))$ in case $\xi \in \tilde{\mathcal{R}}_j$ is close to γ_j , in particular if ξ is close to 0. In this way we find the following behavior as $\xi \rightarrow 0$,

$$\frac{1}{2\pi i} \oint_{\gamma_j} \frac{\log h_j(z(s))}{s - \xi} ds = \begin{cases} c_j + \mathcal{O}(\xi), & \xi \in \mathbb{C} \setminus \tilde{\mathcal{R}}_j \\ c_j - \log h_j(0) + \mathcal{O}(\xi), & \xi \in \tilde{\mathcal{R}}_j \end{cases} \quad (4.43)$$

where

$$c_j = \frac{1}{2\pi i} \oint_{\gamma_j^\varepsilon} \frac{\log h_j(z(s))}{s} ds.$$

By the change of variables $z(s) = x$, $s = \xi_j(z)$, we turn this integral into an integral on a counter that circles around Δ_j in counterclockwise direction. Bringing this integral to Δ_j we obtain

$$c_j = \frac{1}{2\pi i} \int_{\Delta_j} \log h_j(x) \left(\left(\frac{\zeta_j'}{\zeta_j} \right)_- - \left(\frac{\zeta_j'}{\zeta_j} \right)_+ \right) (x) dx. \quad (4.44)$$

Combining (4.42), (4.43), (4.44) we find that for general analytic factors we have

$$\begin{aligned} D_0(z) &= C e^{c_1+c_2} (-p_1(a))^\alpha p_2(a)^\gamma \left(\frac{-p_1(a)p_2(a)}{C(a)^2} \right)^\beta e^{\pm \beta \pi i/3} z^{-2\beta/3} (1 + \mathcal{O}(z^{1/3})), \\ D_1(z) &= C h_1(0) e^{c_1+c_2} (ap_1(a))^\alpha p_2(a)^\gamma \left(\frac{-p_1(a)p_2(a)}{C(a)^2} \right)^\beta z^{\beta/3} (1 + \mathcal{O}(z^{1/3})), \\ D_2(z) &= C h_2(0) e^{c_1+c_2} (-p_1(a))^\alpha p_2(a)^\gamma \left(\frac{-p_1(a)p_2(a)}{C(a)^2} \right)^\beta e^{\mp \beta \pi i/3} z^{\beta/3} (1 + \mathcal{O}(z^{1/3})) \end{aligned} \quad (4.45)$$

as $z \rightarrow 0$ with $\pm \operatorname{Im} z > 0$, with c_1 and c_2 given by (4.44).

For $a = -1$ we have $-p_1(a) = p_2(a) = \frac{1}{\sqrt{3}}$ and $C(a) = 2^{-1/3}$. Since all quantities depend analytically on a , we find from (4.45)

$$\begin{aligned} D_0(z; a) &= C e^{c_1+c_2} \frac{2^{\frac{2}{3}\beta}}{3^{\frac{1}{2}(\alpha+\gamma)+\beta}} e^{\pm \beta \pi i/3} z^{-2\beta/3} (1 + \mathcal{O}(z^{1/3}) + \mathcal{O}(a+1)), \\ D_1(z; a) &= C h_1(0) e^{c_1+c_2} \frac{2^{\frac{2}{3}\beta}}{3^{\frac{1}{2}(\alpha+\gamma)+\beta}} z^{\beta/3} (1 + \mathcal{O}(z^{1/3}) + \mathcal{O}(a+1)), \\ D_2(z; a) &= C h_2(0) e^{c_1+c_2} \frac{2^{\frac{2}{3}\beta}}{3^{\frac{1}{2}(\alpha+\gamma)+\beta}} e^{\mp \beta \pi i/3} z^{\beta/3} (1 + \mathcal{O}(z^{1/3}) + \mathcal{O}(a+1)) \end{aligned} \quad (4.46)$$

as $z \rightarrow 0$ and $a \rightarrow -1$.

4.3.6. Behavior of N around the endpoints. For the further analysis of the Riemann-Hilbert problem for S we need to know the behavior of N around the endpoints $a, 0$ and 1 . To that end we also need to know how the Szegő-functions D_0 , D_1 and D_2 behave around these points.

The functions $D_{0,1,2}(z; w_1, w_2)$ are multiplicative in w_1 and w_2 . Then we can split off the analytical factors h_1, h_2 from the weights w_1, w_2 and write

$$D_j(z; w_1, w_2) = D_j(z; w_1^{(\alpha, \beta)}, w_2^{(\beta, \gamma)}) D_j(z; h_1, h_2)$$

where the functions $D_j(z; w_1^{(\alpha, \beta)}, w_2^{(\beta, \gamma)})$ associated with the weights (4.39) are given in (4.40) above.

The explicit expressions allow us to prove the following proposition:

Proposition 4.2. *Around the branch points N has the following behavior:*

$$N(z) = \begin{cases} \mathcal{O} \begin{pmatrix} (z-a)^{-\frac{1+2\alpha}{4}} & (z-a)^{\frac{2\alpha-1}{4}} & 1 \\ (z-a)^{-\frac{1+2\alpha}{4}} & (z-a)^{\frac{2\alpha-1}{4}} & 1 \\ (z-a)^{-\frac{1+2\alpha}{4}} & (z-a)^{\frac{2\alpha-1}{4}} & 1 \end{pmatrix} & \text{as } z \rightarrow a, \\ \mathcal{O} \begin{pmatrix} (z-1)^{-\frac{1+2\gamma}{4}} & 1 & (z-1)^{\frac{2\gamma-1}{4}} \\ (z-1)^{-\frac{1+2\gamma}{4}} & 1 & (z-1)^{\frac{2\gamma-1}{4}} \\ (z-1)^{-\frac{1+2\gamma}{4}} & 1 & (z-1)^{\frac{2\gamma-1}{4}} \end{pmatrix} & \text{as } z \rightarrow 1, \\ \mathcal{O} \begin{pmatrix} z^{-\frac{2\beta+1}{3}} & z^{\frac{\beta-1}{3}} & z^{\frac{\beta-1}{3}} \\ z^{-\frac{2\beta+1}{3}} & z^{\frac{\beta-1}{3}} & z^{\frac{\beta-1}{3}} \\ z^{-\frac{2\beta+1}{3}} & z^{\frac{\beta-1}{3}} & z^{\frac{\beta-1}{3}} \end{pmatrix} & \text{as } z \rightarrow 0. \end{cases} \quad (4.47)$$

Proof. From the expressions (4.31) and (4.32) together with the behavior of the mapping functions ξ_0, ξ_1, ξ_2 around the branch points we find (4.47) for the case $w_1 \equiv 1, w_2 \equiv 1$, (in which case of course $\alpha = \beta = \gamma = 0$).

For the general case we first note that the functions $D_j(z; h_1, h_2)$ remain bounded and bounded away from 0 for analytic and non-zero h_1 and h_2 . For the Szegő functions (4.40) associated with the pure Jacobi weights we have

$$\begin{aligned} & \begin{pmatrix} D_0(z; w_1^{(\alpha, \beta)}, w_2^{(\beta, \gamma)}) & D_1(z; w_1^{(\alpha, \beta)}, w_2^{(\beta, \gamma)}) & D_2(z; w_1^{(\alpha, \beta)}, w_2^{(\beta, \gamma)}) \end{pmatrix} \\ &= \begin{cases} \mathcal{O} \begin{pmatrix} (z-a)^{-\alpha/2} & (z-a)^{\alpha/2} & 1 \end{pmatrix} & \text{as } z \rightarrow a, \\ \mathcal{O} \begin{pmatrix} (z-1)^{-\gamma/2} & 1 & (z-1)^{\gamma/2} \end{pmatrix} & \text{as } z \rightarrow 1, \\ \mathcal{O} \begin{pmatrix} z^{-2\beta/3} & z^{\beta/3} & z^{\beta/3} \end{pmatrix} & \text{as } z \rightarrow 0, \end{cases} \end{aligned} \quad (4.48)$$

where we use that

$$\xi_j(z) = \mathcal{O}(z^{1/3}) \text{ as } z \rightarrow 0, \quad \text{for } j = 0, 1, 2,$$

$$\xi_j(z) = \mathcal{O}((z-a)^{1/2}) \text{ as } z \rightarrow a, \quad \text{for } j = 0, 1,$$

$$\xi_j(z) = \mathcal{O}((z-1)^{1/2}) \text{ as } z \rightarrow 1, \quad \text{for } j = 0, 2,$$

while $\xi_1(z)$ is analytic around $z = 1$ and $\xi_2(z)$ is analytic around $z = a$. □

4.3.7. Symmetries in the outer parametrix. In this subsection we give two symmetries in the functions \tilde{N} that will be useful later on. Recall that \tilde{N} is the outer parametrix in the case that w_1 and w_2 are identically one, see subsection (4.3.2). A first symmetry deals with the inverse of \tilde{N} .

Proposition 4.3. *For every $a < 0$ we have*

$$\tilde{N}^{-1}(z) = \tilde{N}^T(z), \quad z \in \mathbb{C} \setminus [a, 1]. \quad (4.49)$$

Proof. Define X by

$$X(z) := \tilde{N}(z)\tilde{N}^T(z) \quad z \in \mathbb{C} \setminus [a, 1] \quad (4.50)$$

Using the fact $J_{\tilde{N}} = (J_{\tilde{N}})^{-T}$ we find that on $(a, 0)$ and $(0, 1)$,

$$\begin{aligned} X_-^{-1}X_+ &= \tilde{N}_-^{-T}\tilde{N}_-^{-1}\tilde{N}_+\tilde{N}_+^T = \tilde{N}_-^{-T}J_{\tilde{N}}\tilde{N}_+^T \\ &= \tilde{N}_-^{-T}(J_{\tilde{N}})^{-T}\tilde{N}_+^T = \left(\tilde{N}_+J_{\tilde{N}}^{-1}\tilde{N}_-^{-1}\right)^T = I^T = I \end{aligned} \quad (4.51)$$

Since $\tilde{N}(z)$ tends to I as $z \rightarrow \infty$ we have $X(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \rightarrow \infty$.

By the behavior of \tilde{N} near the branch points (see (4.47) for the case $\alpha = \beta = \gamma = 0$) we obtain that X has no poles in a , 0 or 1 , and we conclude by Liouville's theorem that $X(z) = I$ everywhere and (4.49) follows. \square

As a corollary of this proposition and the expression for the D_j in (4.38) we then also find the behavior of N^{-1} around the branch points, since from (4.34) and (4.49)

$$N^{-1}(z) = D(z)^{-1}\tilde{N}(z)^TD_\infty. \quad (4.52)$$

We then obtain

$$N^{-1}(z) = \begin{cases} \mathcal{O} \begin{pmatrix} (z-a)^{\frac{2\alpha-1}{4}} & (z-a)^{\frac{2\alpha-1}{4}} & (z-a)^{\frac{2\alpha-1}{4}} \\ (z-a)^{-\frac{2\alpha+1}{4}} & (z-a)^{-\frac{2\alpha+1}{4}} & (z-a)^{-\frac{2\alpha+1}{4}} \\ 1 & 1 & 1 \end{pmatrix} & \text{as } z \rightarrow a, \\ \mathcal{O} \begin{pmatrix} (z-1)^{\frac{2\gamma-1}{4}} & (z-1)^{\frac{2\gamma-1}{4}} & (z-1)^{\frac{2\gamma-1}{4}} \\ 1 & 1 & 1 \\ (z-1)^{-\frac{2\gamma+1}{4}} & (z-1)^{-\frac{2\gamma+1}{4}} & (z-1)^{-\frac{2\gamma+1}{4}} \end{pmatrix} & \text{as } z \rightarrow 1, \\ \mathcal{O} \begin{pmatrix} z^{\frac{2\beta-1}{3}} & z^{\frac{2\beta-1}{3}} & z^{\frac{2\beta-1}{3}} \\ z^{-\frac{\beta+1}{3}} & z^{-\frac{\beta+1}{3}} & z^{-\frac{\beta+1}{3}} \\ z^{-\frac{\beta+1}{3}} & z^{-\frac{\beta+1}{3}} & z^{-\frac{\beta+1}{3}} \end{pmatrix} & \text{as } z \rightarrow 0. \end{cases} \quad (4.53)$$

A second symmetry relates the functions \tilde{N} for different values of a to each other. We use $\tilde{N}(\cdot; a)$ to denote the dependence on $a < 0$. Let θ_a be the Möbius transformation leaving 0 and 1 invariant and mapping -1 to a , i.e.,

$$\theta_a(z) = \frac{2az}{(a+1)z+a-1}, \quad \theta_a^{-1}(z) = \frac{(1-a)z}{(a+1)z-2a} \quad (4.54)$$

Proposition 4.4. For $a < 0, a \neq -1$ and any $z \in \mathbb{C} \setminus [a, 1]$ we have

$$\tilde{N}(z; a) = \tilde{N}\left(\frac{1-a}{1+a}; -1\right)^{-1} \tilde{N}(\theta_a^{-1}(z); -1) \quad (4.55)$$

Proof. The proof is similar to the proof of the previous proposition. We do not give details. \square

4.3.8. Behavior at $z = 0$. In the next section we need the leading term for $\tilde{N}(z; a)$ as $z \rightarrow 0$.

Lemma 4.5. As $z \rightarrow 0$ we have

$$\tilde{N}(z; a) = \tilde{N}_0(a) z^{-\frac{1}{3}} + \mathcal{O}(1) \quad (4.56)$$

with

$$\begin{aligned} \tilde{N}_0(a) = \frac{1}{32^{1/6}} \tilde{N}\left(\frac{1-a}{1+a}; -1\right)^{-1} & \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 \end{pmatrix} \\ & \times \begin{cases} \begin{pmatrix} -\omega & 1 & \omega^2 \end{pmatrix} & \text{for } \operatorname{Im} z > 0, \\ \begin{pmatrix} \omega^2 & 1 & \omega \end{pmatrix} & \text{for } \operatorname{Im} z < 0. \end{cases} \end{aligned} \quad (4.57)$$

Proof. By Proposition 4.4 and the fact that $\theta_a^{-1}(z) = z + \mathcal{O}(z^2)$ it suffices to compute the leading term of $\tilde{N}(z; -1)$ as $z \rightarrow 0$. We use the factorization (where we suppress the argument z on the right hand side)

$$\begin{aligned} \tilde{N}(z; -1) = \operatorname{diag} \left(1 \quad \frac{p_1(p_1^2-1)^{1/2}}{p_1-p_2} \quad \frac{p_2(p_2^2-1)^{1/2}}{p_2-p_1} \right) \\ \times \begin{pmatrix} (\xi_0-p_1)(\xi_0-p_2) & (\xi_1-p_1)(\xi_1-p_2) & (\xi_2-p_1)(\xi_2-p_2) \\ \xi_0-p_2 & \xi_1-p_2 & \xi_2-p_2 \\ \xi_0-p_1 & \xi_1-p_1 & \xi_2-p_1 \end{pmatrix} \\ \times \operatorname{diag} \left(\frac{1}{\xi_0(\xi_0^2-1)^{\frac{1}{2}}} \quad \frac{1}{\xi_1(\xi_1^2-1)^{\frac{1}{2}}} \quad \frac{1}{\xi_2(\xi_2^2-1)^{\frac{1}{2}}} \right), \end{aligned} \quad (4.58)$$

A careful analysis of all the functions and constants involved then shows that for $\operatorname{Im} z > 0$

$$\tilde{N}(z; -1) = \frac{1}{32^{1/6}} z^{-\frac{1}{3}} \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} -\omega & 1 & \omega^2 \end{pmatrix} + \mathcal{O}(1) \text{ as } z \rightarrow 0, \quad (4.59)$$

and for $\operatorname{Im} z < 0$

$$\tilde{N}(z; -1) = \frac{1}{32^{1/6}} z^{-\frac{1}{3}} \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} \omega^2 & 1 & \omega \end{pmatrix} + \mathcal{O}(1) \text{ as } z \rightarrow 0. \quad (4.60)$$

Together with (4.55) this proves the lemma. \square

4.4. Local parametrices

4.4.1. Local parametrices around ± 1 . The outer parametrix N is intended as an approximation to S . However the approximation cannot be good around the branch points a , 0 and 1 . Indeed, the entries of $S(z)N^{-1}(z)$ will typically diverge as z tends to one of the branch points.

The solution to this problem is building local approximations around the branch points, called local parametrices. The appropriate construction around a and 1 is standard, and uses the Bessel model parametrix as defined in [29], equations (6.23)-(6.25). Let U_{-1} and U_1 be disks around respectively -1 and 1 of fixed but small enough radius: U_1 should be contained in V_2 , such that w_2 is well-defined on $U_1 \setminus [1, +\infty)$. Similarly we must have $U_{-1} \subset V_1$, and additionally a must lie inside U_{-1} . On these disks we construct 3×3 matrix valued functions P_{-1} and P_1 that satisfy the same jumps as S , see (4.16)-(4.17), and match with N on the boundary of the disks:

$$\begin{aligned} P_{-1}(z)N(z)^{-1} &= I + \mathcal{O}\left(\frac{1}{n}\right) & \text{for } z \in \partial U_{-1}, \\ P_1(z)N(z)^{-1} &= I + \mathcal{O}\left(\frac{1}{n}\right) & \text{for } z \in \partial U_1, \end{aligned} \quad (4.61)$$

as $n \rightarrow \infty$. The \mathcal{O} -terms are uniform in z . For details of the construction of P_{-1} and P_1 we refer to [29], where the Bessel model parametrix was introduced, and [31], where it was also used in a 3×3 matrix valued Riemann-Hilbert problem.

4.4.2. Local parametrix around 0: statement. Around 0 we need a new kind of local parametrix P_0 on a disk U_0 around the origin. There are a number of difficulties to obtain the desired matching condition

$$P_0(z)N(z)^{-1} = I + \mathcal{O}(n^{-\kappa}) \quad \text{for } z \in \partial U_0$$

with some $\kappa > 0$, that in fact we are unable to resolve. The best we can do is to construct P_0 such that $P_0(z)N(z)^{-1}$ remains bounded as $n \rightarrow \infty$ for z on a circle of radius that decays like $n^{-1/2}$ as $n \rightarrow \infty$.

Thus the disk U_0 should be shrinking as n increases, and for definiteness we take

$$\text{radius } U_0 := n^{-\frac{1}{2}}, \quad (4.62)$$

and we assume n to be large enough so that U_0 is contained in $V_1 \cap V_2$. Then consider the following Riemann-Hilbert problem for the local parametrix P_0 around 0 .

- P_0 is analytic on $U_0 \setminus \Sigma_S$,
- P_0 has jumps

$$P_{0,+} = P_{0,-} J_P \quad \text{on } \Sigma_S \cap U_0, \quad \text{where } J_P = J_S, \quad (4.63)$$

see (4.16)-(4.17),

- $P_0(z)$ behaves in the same way as $S(z)$ as $z \rightarrow 0$, see (4.20),

- $P_0 N^{-1}$ remains bounded on the boundary of U_0 ,

$$P_0(z)N(z)^{-1} = \mathcal{O}(1) \quad \text{for } z \in \partial U_0, \quad (4.64)$$

as $n \rightarrow \infty$, where $a = a_n$ depends on n as in (1.9).

Notice that the matching between $P_0(z)$ and $N(z)$ does not improve with increasing n . Indeed, the matrix $P_0(z)N(z)^{-1}$ does not tend to I as $n \rightarrow \infty$ for $z \in \partial U_0$. The matching (4.64) is the best we can obtain without modifying the outer parametrix N . However, with $a = a_n$ as in (1.9) we will be able to find a 3×3 matrix valued function $Z_n(\cdot; a)$ such that

$$P_0(z)N(z)^{-1} = I + Z_n(z; a) + \mathcal{O}\left(n^{-1/6}\right) \quad \text{for } z \in \partial U_0 \quad (4.65)$$

The explicit expression and special properties of Z_n will allow us to create, in the final transformation, a jump on ∂U_0 that tends to I as $n \rightarrow \infty$.

Since the dependence on a will be important, we emphasize that most notions depend on a and have limiting values as $a \rightarrow -1$. As before, we will not always explicitly indicate the dependence on a , but sometimes we do.

4.4.3. Reduction to constant jumps. We factor out the φ_i and w_i -functions from the jump matrices (4.16)-(4.17). Define for $z \in U_0$ the matrix valued functions

$$\Lambda(z) := \frac{2}{3} \begin{pmatrix} \varphi_1(z) + \varphi_2(z) & 0 & 0 \\ 0 & \varphi_2(z) - 2\varphi_1(z) & 0 \\ 0 & 0 & \varphi_1(z) - 2\varphi_2(z) \end{pmatrix}, \quad (4.66)$$

$$W(z) := \begin{pmatrix} z^\beta & 0 & 0 \\ 0 & (z-a)^{-\alpha} h_1(z)^{-1} & 0 \\ 0 & 0 & (1-z)^{-\gamma} h_2(z)^{-1} \end{pmatrix}. \quad (4.67)$$

We look for P_0 in the form

$$P_0(z) = \tilde{P}_0(z) e^{n\Lambda(z)} W^{-1}(z). \quad (4.68)$$

In order that P_0 has the jumps J_P , we should have $\tilde{P}_{0,+} = \tilde{P}_{0,-} J_{\tilde{P}}$ with

$$J_{\tilde{P}} = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ e^{\pm\beta\pi i} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } \Sigma_1^\pm \cap U_0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} & \text{on } \Sigma_2^\pm \cap U_0, \\ \begin{pmatrix} 0 & e^{\beta\pi i} & 0 \\ -e^{\beta\pi i} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } [a, 0] \cap U_0, \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} & \text{on } [0, 1] \cap U_0. \end{cases} \quad (4.69)$$

4.4.4. Functions $f(z)$ and $\tau(z)$. Note that the jumps (4.69) are exactly the same as the ones for Ψ , see Figure 3, except that the jumps for Ψ are on unbounded rays. Recall that $\Psi(z; \tau)$ also depends on τ which appears in the asymptotic condition (2.9).

Our aim is to construct \tilde{P}_0 of the form

$$\tilde{P}_0(z) = E_n(z) \Psi \left(n^{\frac{3}{2}} f(z); n^{\frac{1}{2}} \tau(z) \right), \quad (4.70)$$

where $f(z)$ is a conformal map and $\tau(z)$ is analytic in U_0 . The matrix valued function $E_n(z)$ is an analytic prefactor, which will be defined in the next subsection. We are going to choose $f(z)$ and $\tau(z)$ such that

$$\Theta \left(n^{\frac{3}{2}} f(z); n^{\frac{1}{2}} \tau(z) \right) + n\Lambda(z) = 0 \quad \text{for } z \in U_0 \setminus \mathbb{R} \quad (4.71)$$

where Θ is given by (2.9) and Λ is given by (4.66). When this condition is satisfied there will be no exponential growth (as $n \rightarrow \infty$) in $P_0(z)$ see (4.68), and so there is a chance that we can match it with N .

Define functions $\lambda_1(z)$ and $\lambda_2(z)$ on $\mathbb{C} \setminus \mathbb{R}$ by

$$\lambda_1(z) := \begin{cases} -z^{-\frac{1}{3}} (\varphi_1(z) + \omega^2 \varphi_2(z)) & \text{for } \operatorname{Im} z > 0, \\ -z^{-\frac{1}{3}} (\varphi_1(z) + \omega \varphi_2(z)) & \text{for } \operatorname{Im} z < 0, \end{cases} \quad (4.72)$$

$$\lambda_2(z) := \begin{cases} -z^{-\frac{2}{3}} (\varphi_1(z) + \omega \varphi_2(z)) & \text{for } \operatorname{Im} z > 0, \\ -z^{-\frac{2}{3}} (\varphi_1(z) + \omega^2 \varphi_2(z)) & \text{for } \operatorname{Im} z < 0, \end{cases} \quad (4.73)$$

where φ_1 and φ_2 are given by (4.5). It can be checked that the functions $\lambda_1(z)$ and $\lambda_2(z)$ have no jumps on $(a, 0)$ or $(0, 1)$. Since φ_1 and φ_2 are bounded, λ_1 and λ_2 have analytic continuations to $\mathbb{C} \setminus ((-\infty, a] \cup [1, \infty))$. Since the φ -functions depend on a , so do the λ -functions, and we write $\lambda(z; a)$ to emphasize this fact.

It may be checked that $\lambda_1(z; a)$ and $\lambda_2(z; a)$ converge uniformly in a neighborhood of zero as $a \rightarrow -1$, and also that

$$\begin{aligned}\lambda_1(0; a) &= \frac{3 \cdot 2^{2/3}}{4}(a+1) + \mathcal{O}(a+1)^2, \\ \lambda_2(0; a) &= \frac{9 \cdot 2^{1/3}}{4} + \mathcal{O}(a+1)\end{aligned}\tag{4.74}$$

as $a \rightarrow -1$. Then $\operatorname{Re} \lambda_2(z; a) > 0$ for a close enough to -1 and $z \in U_0$, and we can define the following analytic functions in a neighborhood of 0.

Definition 4.6. For a close enough to -1 and $z \in U_0$ we define

$$\begin{aligned}f(z) &= f(z; a) := \frac{8}{27} z \lambda_2(z; a)^{\frac{3}{2}}, \\ \tau(z) &= \tau(z; a) := \frac{\lambda_1(z; a)}{\lambda_2(z; a)^{\frac{1}{2}}}.\end{aligned}\tag{4.75}$$

Then f is a conformal map with $f(0) = 0$, and $f(z)$ is real for real arguments z . By (4.74) and (4.75) we have as $a \rightarrow -1$.

$$\begin{aligned}f'(0; a) &= \sqrt{2} + \mathcal{O}(a+1), \\ \tau(0; a) &= \frac{1}{\sqrt{2}}(a+1) + \mathcal{O}(a+1)^2.\end{aligned}\tag{4.76}$$

Without loss of generality we can now assume that the lips of the lenses are chosen such that f maps $\Sigma_S \cap U_0$ into Σ_Ψ . Then $\Psi\left(n^{\frac{3}{2}}f(z); n^{\frac{1}{2}}\tau(z)\right)$ is well-defined and analytic in $U_0 \setminus \Sigma_S$. It remains to check the condition (4.71). By (2.10) and (4.75) we find:

$$\begin{aligned}\theta_k\left(n^{\frac{3}{2}}f(z); n^{\frac{1}{2}}\tau(z)\right) &= -\frac{3n}{2}\omega^k f(z)^{\frac{2}{3}} - n\tau(z)\omega^{2k} f(z)^{\frac{1}{3}} \\ &= -\frac{2n}{3}\left(\omega^k z^{\frac{2}{3}}\lambda_2(z) + \omega^{2k} z^{\frac{1}{3}}\lambda_1(z)\right)\end{aligned}\tag{4.77}$$

By (4.72) and (4.73) the right hand sides are exactly minus n times the components of Λ , and (4.71) follows.

4.4.5. Prefactor $E_n(z)$. Next we define a suitable analytic prefactor $E_n(z) = E_n(z; a)$ such that the local parametrix

$$P_0(z) = E_n(z)\Psi\left(n^{\frac{3}{2}}f(z); n^{\frac{1}{2}}\tau(z)\right)e^{n\Lambda(z)}W(z)^{-1}\tag{4.78}$$

satisfies the matching condition (4.64) with N on ∂U_0 . Also we obtain an expression for the function Z_n in (4.65).

Denote by $A(z; \tau)$ the right-hand side of (2.7) without the exponential factor $e^{\Theta(z; \tau)}$ and the error factor $I + \mathcal{O}(z^{-\frac{1}{3}})$. Thus

$$A(z; \tau) := \sqrt{\frac{2\pi}{3}} e^{\tau^2/6} z^{\frac{\beta}{3}} \begin{pmatrix} z^{\frac{1}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-\frac{1}{3}} \end{pmatrix} \Omega_{\pm} B_{\pm}, \quad \pm \operatorname{Im} z > 0.\tag{4.79}$$

Then we define

Definition 4.7. For a close enough to -1 and $z \in U_0$ we define

$$E_n(z; a) = N(z; a)W(z; a)A^{-1} \left(n^{\frac{3}{2}}f(z; a); n^{\frac{1}{2}}\tau(z; a) \right). \quad (4.80)$$

where $f(z; a)$ and $\tau(z; a)$ are given by (4.75).

Proposition 4.8. *The function E_n defined by (4.80) is analytic on U_0 .*

Proof. The function A from (4.79) has jumps on the real line, given by $A_+ = A_- J_A$ with

$$J_A = \begin{pmatrix} 0 & e^{\beta\pi i} & 0 \\ -e^{\beta\pi i} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ on } \mathbb{R}^-, \quad J_A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ on } \mathbb{R}^+. \quad (4.81)$$

Then $A \left(n^{\frac{3}{2}}f(z); n^{\frac{1}{2}}\tau(z) \right)$ has the corresponding jumps on $\mathbb{R} \cap U_0$. A straightforward calculation reveals that $N(z)W(z)$ has exactly the same jumps on $U_0 \cap (a, 0)$ and $U_0 \cap (0, 1)$, and hence $E_n(z)$ has no branch cuts in U_0 .

Also E_n can have no pole in 0. By (4.47) and the definition of W (4.67) we have that

$$N(z)W(z) = \mathcal{O} \left(z^{\frac{\beta-1}{3}} \right) \text{ as } z \rightarrow 0 \quad (4.82)$$

From (4.79) and the fact that f is a conformal map with $f(0) = 0$ we obtain

$$A \left(n^{\frac{3}{2}}f(z); n^{\frac{1}{2}}\tau(z) \right)^{-1} = \mathcal{O} \left(z^{-\frac{\beta+1}{3}} \right) \quad (4.83)$$

Then by (4.80) and (4.82) and (4.83) we see that $E_n(z) = \mathcal{O}(z^{-\frac{2}{3}})$, and therefore the isolated singularity at 0 is removable. \square

We remark that by a similar argument we have that $E_n^{-1}(z; a)$ is analytic in U_0 and in particular at $z = 0$ as well.

4.4.6. Matching condition. We show that the matching condition (4.64) holds, and we compute Z_n from (4.65).

Proposition 4.9. *The parametrix P_0 defined by (4.68), (4.70), (4.80) satisfies the matching condition (4.64) as $n \rightarrow \infty$ with $a = a_n$ as in (1.9). The matching (4.65) holds with*

$$\begin{aligned} Z_n(z; a) = & -\frac{1}{81 \cdot 2^{1/3}} \frac{\tau(z; a)(n\tau^2(z; a) + 9\beta)}{z^{\frac{2}{3}}f(z; a)^{\frac{1}{3}}} \\ & \times D_\infty^{-1} \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 \end{pmatrix} (\sqrt{2}i \quad 1 \quad -1) D_\infty. \end{aligned} \quad (4.84)$$

Proof. The local parametrix P_0 , written in full, is given by

$$P_0(z) = N(z)W(z)A\left(n^{\frac{3}{2}}f(z); n^{\frac{1}{2}}\tau(z)\right)^{-1}\Psi\left(n^{\frac{3}{2}}f(z), n^{\frac{1}{2}}\tau(z)\right)e^{n\Lambda(z)}W(z)^{-1}, \quad (4.85)$$

where all functions also depend on a .

We first observe that $n^{\frac{1}{2}}\tau(z; a)$ remains bounded for $z \in U_0$ as $n \rightarrow \infty$. Here we need the fact that we took the radius of U_0 to be $n^{-\frac{1}{2}}$, and that $a = a_n = -1 + \mathcal{O}(n^{-1/2})$. Since $\tau(0; -1) = 0$, see (4.76) we indeed obtain

$$\tau(z; a) = \tau(z; -1) + \mathcal{O}(a - a_n) = \mathcal{O}(n^{-1/2}) \quad \text{for } |z| = n^{-1/2}, \quad (4.86)$$

as $n \rightarrow \infty$.

It follows that we can use the asymptotic expansion (2.22) for $\Psi(z; \tau)$ in (4.85), since (2.22) is uniformly valid for τ -values in a bounded set. Then by combining (2.22) and the definition (4.79) of A we get that a number of factors cancel, and what remains from (4.85) is

$$P_0(z)N(z)^{-1} = N(z)W(z)B_{\pm}^{-1} \times \left(I + \frac{(\Psi_1)_{\pm}(n^{\frac{1}{2}}\tau(z))}{n^{\frac{1}{2}}f(z)^{\frac{1}{3}}} + \mathcal{O}(n^{-\frac{2}{3}}) \right) B_{\pm}W(z)^{-1}N(z)^{-1} \quad (4.87)$$

as $n \rightarrow \infty$ uniformly for $z \in \partial U_0$.

For $z \in \partial U_0$ we have $|z| = n^{-\frac{1}{2}}$ and the entries of NW are $\mathcal{O}(z^{\frac{\beta-1}{3}})$ as $z \rightarrow 0$. Hence $(NW)(z)$ is $\mathcal{O}(n^{\frac{1-\beta}{6}})$ for z on ∂U_0 as $n \rightarrow \infty$. For the inverse of NW we get $\mathcal{O}(n^{-\frac{\beta+1}{6}})$ entries on ∂U_0 . Hence the $\mathcal{O}(n^{-\frac{2}{3}})$ term in the middle factor of the right hand side of (4.87) turns into a $\mathcal{O}(n^{-\frac{1}{3}})$ term:

$$P_0(z)N(z)^{-1} = I + \frac{1}{n^{\frac{1}{2}}f(z)^{\frac{1}{3}}}N(z)W(z)B_{\pm}^{-1} \times (\Psi_1)_{\pm}(n^{\frac{1}{2}}\tau(z))B_{\pm}W(z)^{-1}N(z)^{-1} + \mathcal{O}\left(n^{-\frac{1}{3}}\right). \quad (4.88)$$

In the same way we find that the second term in the right-hand side of (4.88) is $\mathcal{O}(1)$ as $n \rightarrow \infty$. We evaluate this term in more detail.

By (4.34) we have

$$N(z)W(z)B_{\pm}^{-1} = D_{\infty}^{-1}\tilde{N}(z)D(z)W(z)B_{\pm}^{-1}$$

where by (4.33)–(4.34), (4.67), and (2.8), the last three matrices on the right-hand side are diagonal and their product satisfies for some constant $c \neq 0$,

$$D(z)W(z)B_{\pm}^{-1} = cz^{\beta/3}(I + \mathcal{O}(z)) \quad \text{as } z \rightarrow 0.$$

Furthermore the leading behavior of $\tilde{N}(z)$ and its inverse as $z \rightarrow 0$ follow from (4.49) and (4.56). Thus (4.88) reduces to (where we emphasize again the dependence on a),

$$P_0(z)N(z)^{-1} = I + \frac{1}{n^{\frac{1}{2}}z^{\frac{2}{3}}f(z;a)^{\frac{1}{3}}} \times D_\infty^{-1}\tilde{N}_0(a)(\Psi_1)_\pm(n^{\frac{1}{2}}\tau(z;a))\tilde{N}_0^T(a)D_\infty + \mathcal{O}\left(n^{-\frac{1}{6}}\right). \quad (4.89)$$

By (4.57) and (4.30) we have

$$\tilde{N}_0(a) = \frac{1}{3 \cdot 2^{1/6}} (I + \mathcal{O}(a+1)) \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 \end{pmatrix} \times \begin{cases} \begin{pmatrix} -\omega & 1 & \omega^2 \end{pmatrix} & \text{for } \operatorname{Im} z > 0, \\ \begin{pmatrix} \omega^2 & 1 & \omega \end{pmatrix} & \text{for } \operatorname{Im} z < 0, \end{cases}$$

as $a \rightarrow -1$. We plug this and the explicit formulas for $(\Psi_1)_\pm$ (see Lemma 2.3) into (4.89). Then after some calculations we indeed obtain (4.65) with $Z_n(z;a)$ given by (4.84).

Since $n\tau^2(z;a)$ remains bounded as $n \rightarrow \infty$ if $|z| = n^{-1/2}$ and $a = a_n = -1 + \mathcal{O}(n^{-1/2})$, we obtain that $Z_n(z;a) = \mathcal{O}(1)$. This proves the proposition. \square

We note from (4.84) that $Z_n(z;a)$ is analytic in a punctured neighborhood of $z = 0$ with a simple pole at $z = 0$. It also follows from (4.84) that

$$Z_n(z_1;a)Z_n(z_2;a) = 0 \quad \text{for all } z_1, z_2 \text{ near } 0. \quad (4.90)$$

This property will be important in the final transformations

5. Final transformations

We will do the final transformation $S \mapsto R$ in two steps. First we will define R_0 as the approximation error between S and the parametrices N, P_{-1}, P_0 and P_1 . The jump matrices of R_0 will tend to the identity matrix as $n \rightarrow \infty$ on all parts of the jump contour Σ_R , except on ∂U_0 .

Via a global transformation we finally define R , in such a way that it also has a jump that tends to the identity matrix on U_0 . As a result of the steepest descent analysis, we then derive a global and uniform estimate for this function R .

5.1. Transformation $S \mapsto R_0$

Define $R_0(z)$ as

$$R_0(z) = \begin{cases} S(z)N^{-1}(z), & z \in \mathbb{C} \setminus (\Sigma_S \cup \overline{U_{-1}} \cup \overline{U_0} \cup \overline{U_1}), \\ S(z)P_{-1}^{-1}(z), & z \in U_{-1} \setminus \Sigma_S, \\ S(z)P_0^{-1}(z), & z \in U_0 \setminus \Sigma_S, \\ S(z)P_1^{-1}(z), & z \in U_1 \setminus \Sigma_S. \end{cases} \quad (5.1)$$

Then R_0 is defined and analytic on \mathbb{C} minus the interval $[a, 1]$, the lenses $\Sigma_{1,2}^\pm$, and the circles ∂U_{-1} , ∂U_0 , and ∂U_1 . By comparing the jumps of S and

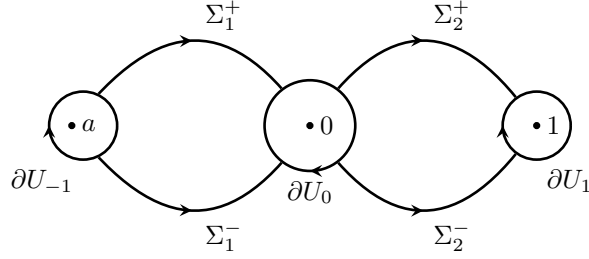


FIGURE 9. The contour Σ_R that consists of the circles ∂U_{-1} , ∂U_0 , ∂U_1 and the parts of the lenses Σ_1^\pm and Σ_2^\pm outside the disks.

the parametrices we find that R has analytic continuation into each of the disks, and across the parts of the real intervals $(a, 0)$ and $(0, 1)$ outside of the disks. The singularities at a , 0 and 1 are removable. For a and 1 this follows from the behavior of the Bessel parametrix given in [29]. For 0 it requires a special check involving the behavior of Ψ and Ψ^{-1} that we will not give here. The function R_0 will then have jumps along the reduced contour Σ_R shown in Figure 9.

We choose clockwise orientation for the circles. The lips of the lenses are oriented from left to right, as before. Then R_0 satisfies the following RH problem.

- R_0 is defined and analytic on $\mathbb{C} \setminus \Sigma_R$,
- R_0 satisfies the jump relation $R_{0,+} = R_{0,-} J_{R_0}$ on Σ_R with

$$J_{R_0}(z) = \begin{cases} N(z)J_S(z)N^{-1}(z), & z \in \Sigma_R \setminus (\partial U_{-1} \cup \partial U_0 \cup \partial U_1), \\ P_{-1}(z)N^{-1}(z), & z \in \partial U_{-1}, \\ P_0(z)N^{-1}(z), & z \in \partial U_0, \\ P_1(z)N^{-1}(z), & z \in \partial U_1, \end{cases} \quad (5.2)$$

- $R_0(z) = I + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$.

Due to the matching conditions for the local parametrices (4.61) and (4.64) we have

$$J_{R_0}(z) = I + \mathcal{O}\left(\frac{1}{n}\right) \text{ uniformly for } z \in \partial U_{-1} \cup \partial U_1 \quad (5.3)$$

On the lips of the lenses the off-diagonal entries of the jumps of S involve the functions $\varphi_{1,2}$, and they do not necessarily have negative real parts along the lenses $\Sigma_{1,2}^\pm$. However, due to the estimates in Lemma 4.1 we may conclude that for $a = a_n = -1 + \mathcal{O}(n^{-1/2})$, we have for some constant $C > 0$,

$$\operatorname{Re} \varphi_j(z; a_n) < -Cn^{-1/3}, \quad \text{for } z \in \Sigma_R \setminus (\partial U_{-1} \cup \partial U_0 \cup \partial U_1)$$

Then by (4.17) we have $J_S = I + \mathcal{O}(e^{-Cn^{2/3}})$, and so by (5.2) we also obtain

$$J_{R_0}(z) = I + \mathcal{O}\left(e^{-Cn^{2/3}}\right) \text{ as } n \rightarrow \infty \quad (5.4)$$

with a possibly different constant $C > 0$. The \mathcal{O} -term in (5.4) holds uniformly for $z \in \Sigma_R \setminus (\partial U_{-1} \cup \partial U_0 \cup \partial U_1)$.

Due to (4.65) and (5.2) the jump matrix J_{R_0} on ∂U_0 is given by

$$J_{R_0}(z) = P_0(z)N(z)^{-1} = I + Z_n(z; a) + \mathcal{O}\left(n^{-\frac{1}{6}}\right) \text{ for } z \in \partial U_0, \quad (5.5)$$

with $Z_n(z; a)$ given by (4.84), and the \mathcal{O} -term is valid under the assumption that $a = a_n = -1 + \mathcal{O}(n^{-1/2})$ as $n \rightarrow \infty$, see Proposition 4.9. The jump matrix on ∂U_0 does not converge to the identity matrix as $n \rightarrow \infty$. Therefore we need one more transformation.

5.2. Transformation $R_0 \mapsto R$

Let $Z_n^{(0)}(a)$ denote the residue of $Z_n(z; a)$ at the simple pole $z = 0$. Thus by (4.84)

$$\begin{aligned} Z_n^{(0)}(a) &= \lim_{z \rightarrow 0} z Z_n(z; a) = -\frac{1}{81 \cdot 2^{1/3}} \frac{\tau(0; a)(n\tau^2(0; a) + 9\beta)}{f'(0; a)^{1/3}} \\ &\quad \times D_\infty^{-1} \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 \end{pmatrix} (\sqrt{2}i \quad 1 \quad -1) D_\infty. \end{aligned} \quad (5.6)$$

We define the new matrix-valued function R as

$$R(z) = R_0(z) \times \begin{cases} I - \frac{Z_n^{(0)}(a)}{z}, & \text{for } z \in \mathbb{C} \setminus (\Sigma_R \cup \overline{U}_0), \\ I + Z_n(z; a) - \frac{Z_n^{(0)}(a)}{z}, & \text{for } z \in U_0. \end{cases} \quad (5.7)$$

Then R is defined and analytic in $\mathbb{C} \setminus \Sigma_R$, and satisfies

- R is analytic on $\mathbb{C} \setminus \Sigma_R$,
- R satisfies the jump relation $R_+ = R_- J_R$ on Σ_R with

$$J_R(z) = \begin{cases} \left(I + Z_n(z; a) - \frac{Z_n^{(0)}(a)}{z} \right)^{-1} J_{R_0}(z) \left(I + \frac{Z_n^{(0)}(a)}{z} \right) & z \in \partial U_0, \\ \left(I - \frac{Z_n^{(0)}(a)}{z} \right)^{-1} J_{R_0}(z) \left(I - \frac{Z_n^{(0)}(a)}{z} \right) & z \in \Sigma_R \setminus \partial U_0. \end{cases} \quad (5.8)$$

- $R(z) = I + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$.

Note that by (4.84) and (5.6) both $Z_n(z; a)$ and $Z_n^{(0)}(a)$ are scalar multiples of the constant matrix $D_\infty^{-1} (\sqrt{2}i \quad 1 \quad -1)^T (\sqrt{2}i \quad 1 \quad -1) D_\infty$ whose square is zero. Therefore we have such relations as

$$\left(I + Z_n(z; a) - \frac{Z_n^{(0)}(a)}{z} \right)^{-1} = I - Z_n(z; a) + \frac{Z_n^{(0)}(a)}{z} \quad (5.9)$$

and $Z_n(z; a)^2 = Z_n(z; a)Z_n^{(0)}(a) = Z_n^{(0)}(a)Z_n(z; a) = (Z_n^{(0)}(a))^2 = 0$.

We also recall that for $a = a_n$ as in (1.9) we have by (4.76) that $n^2\tau(0; a_n)$ remains bounded as $n \rightarrow \infty$, while $f'(0, a_n)$ tends to $\sqrt{2}$. Therefore by (5.6)

$$Z_n^{(0)}(a_n) = \mathcal{O}(n^{-1/2}) \quad \text{as } n \rightarrow \infty. \quad (5.10)$$

This implies by (5.2) that the same estimates as we had in (5.3), (5.4) for J_{R_0} remain valid for J_R on $\Sigma_R \setminus \partial U_0$. That is,

$$J_R(z) = \begin{cases} I + \mathcal{O}(n^{-1}) & \text{for } z \in \partial U_{-1} \cup \partial U_1, \\ I + \mathcal{O}\left(e^{-Cn^{2/3}}\right) & \text{for } z \in \Sigma_R \setminus (\partial U_{-1} \cup \partial U_0 \cup \partial U_1). \end{cases} \quad (5.11)$$

From (4.65), (5.2), (5.8), and (5.9) we have when $a = a_n$ as in (1.9),

$$\begin{aligned} J_R(z) &= \left(I - Z_n(z; a_n) + \frac{Z_n^{(0)}(a_n)}{z} \right) \\ &\quad \times \left(I + Z_n(z; a_n) + \mathcal{O}\left(n^{-\frac{1}{6}}\right) \right) \left(I - \frac{Z_n^{(0)}(a_n)}{z} \right) \\ &= I + \mathcal{O}\left(n^{-\frac{1}{6}}\right), \quad z \in \partial U_0, \end{aligned} \quad (5.12)$$

where we also used (5.10) and the fact that $Z_n(z; a_n)$ remains bounded for $|z| = n^{-1/2}$.

5.3. Conclusion of the steepest descent analysis

We have now reached the goal of the steepest descent analysis. In the RH problem for R we have by (5.11) and (5.12) that all jump matrices J_R tend to the identity matrix as $n \rightarrow \infty$.

Then by standard methods, see e.g. [14] and also [10] for a situation with varying contours, we have that $R(z)$ also tends to the identity matrix as $n \rightarrow \infty$, at the following rate

$$R(z) = I + \mathcal{O}\left(\frac{n^{-\frac{1}{6}}}{1 + |z|}\right) \text{ uniformly for } z \in \mathbb{C} \setminus \Sigma_R. \quad (5.13)$$

This concludes the steepest descent analysis of the Riemann-Hilbert problem for Y .

6. Proof of Theorem 1.4

Recall that $P_{n,n}(z; a)$ is the $(1, 1)$ entry of Y , see (2.6), which we write as

$$P_{n,n}(z; a) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} Y(z) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (6.1)$$

The asymptotic formula (1.10) for $P_{n,n}$ will be derived from this by following the series of transformations $Y \mapsto T \mapsto S \mapsto R$ for $z \in U_0$.

In the calculations that follow we assume that z is in the upper part of the lens around $[0, 1]$. The proof for other z in other regions is similar. We obtain from

(4.4), (4.14), and (6.1)

$$P_{n,n}(z; a) = e^{2n(g_1(z)+g_2(z))} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} S(z) \begin{pmatrix} 1 \\ 0 \\ w_2(z)^{-1} e^{2n\varphi_2(z)} \end{pmatrix}. \quad (6.2)$$

For $z \in U_0$ we have by (4.78) and (5.1)

$$S(z) = R_0(z)P_0(z) = R_0(z)E_n(z; a)\Psi\left(n^{\frac{3}{2}}f(z; a); n^{\frac{1}{2}}\tau(z; a)\right) e^{n\Lambda(z)}W(z)^{-1}.$$

Inserting this into (6.2) and using the expressions (4.66) and (4.67) for Λ and W we obtain

$$P_{n,n}(z; a) = z^{-\beta} e^{2n(g_1(z)+g_2(z))} e^{\frac{2}{3}n(\varphi_1(z)+\varphi_2(z))} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} R_0(z)E_n(z; a) \\ \times \Psi\left(n^{\frac{3}{2}}f(z; a); n^{\frac{1}{2}}\tau(z; a)\right) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (6.3)$$

where we used $w_2(z) = z^\beta(1-z)^\gamma h_2(z)$, see (4.12). The scalar prefactor in (6.3) simplifies because of (4.5) and the result is

$$P_{n,n}(z; a) = (-1)^n e^{-\frac{2n}{3}(l_1+l_2)} z^{-\beta} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} R_0(z)E_n(z; a) \\ \times \Psi\left(n^{\frac{3}{2}}f(z; a); n^{\frac{1}{2}}\tau(z; a)\right) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (6.4)$$

Note that by (4.34), (4.80), (4.84), and (5.7)

$$R_0(z)E_n(z; a) = R(z) \left(I + Z_n(z; a) - \frac{Z_n^{(0)}(a)}{z} \right) D_\infty^{-1} \tilde{N}(z; a) \\ \times D(z)W(z)A \left(n^{\frac{3}{2}}f(z; a); n^{\frac{1}{2}}\tau(z; a) \right)^{-1} \quad (6.5)$$

Assuming that $a = -1 + \mathcal{O}(n^{-1/2})$ and $z = \mathcal{O}(n^{-3/2})$ as $n \rightarrow \infty$, we have by (4.56) and (4.57)

$$\tilde{N}(z) = \frac{1}{3 \cdot 2^{1/6}} z^{-\frac{1}{3}} \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} -\omega & 1 & \omega^2 \end{pmatrix} (I + \mathcal{O}(n^{-1/2})). \quad (6.6)$$

Using this in the product (6.5) we see that the terms $Z_n(z; a) - \frac{Z_n^{(0)}(a)}{z}$ are canceled by the leading order term in (6.6) because of the special form (4.84) of $Z_n(z; a)$.

Since $Z_n(z; a) - \frac{Z_n^{(0)}(a)}{z}$ is uniformly bounded, we find that the first four factors on the right-hand side of (6.5) combine to (we also use (5.13)),

$$\left(I + \mathcal{O}(n^{-1/6}) \right) \frac{1}{3 \cdot 2^{1/6}} z^{-\frac{1}{3}} D_\infty^{-1} \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} -\omega & 1 & \omega^2 \end{pmatrix}. \quad (6.7)$$

The behavior of $D(z)$ is determined by (4.46). Then by (4.67)

$$D(z)W(z) = Ce^{c_1+c_2} \frac{2^{\frac{2}{3}\beta}}{3^{\frac{1}{2}(\alpha+\gamma)+\beta}} z^{\beta/3} B_+(I + \mathcal{O}(n^{-1/2}))$$

where B_+ is the diagonal matrix given by (2.8). This factor also appears in the definition (4.79) of $A(z; \tau)$. Then the product of the last three factors on the right-hand side of (6.5) gives us

$$\begin{aligned} & Ce^{c_1+c_2} e^{-\frac{1}{6}n\tau(z;a)^2} \sqrt{\frac{3}{2\pi}} \frac{2^{\frac{2}{3}\beta}}{3^{\frac{1}{2}(\alpha+\gamma)+\beta}} \left(\frac{n^{\frac{3}{2}}f(z;a)}{z} \right)^{-\frac{\beta}{3}} \\ & \times \Omega_+^{-1} \begin{pmatrix} \left(n^{\frac{3}{2}}f(z;a) \right)^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \left(n^{\frac{3}{2}}f(z;a) \right)^{1/3} \end{pmatrix} (I + \mathcal{O}(n^{-1/2})) \end{aligned} \quad (6.8)$$

Since $(-\omega \ 1 \ \omega^2) \Omega_+^{-1} = (0 \ 0 \ 1)$, we obtain by multiplying (6.7) and (6.8),

$$\begin{aligned} R_0(z)E_n(z;a) &= \frac{Ce^{c_1+c_2} e^{-\frac{1}{6}n\tau(z;a)^2}}{\sqrt{6\pi} \cdot 2^{1/6}} \frac{2^{\frac{2}{3}\beta}}{3^{\frac{1}{2}(\alpha+\gamma)+\beta}} \left(\frac{n^{\frac{3}{2}}f(z;a)}{z} \right)^{\frac{1}{3}-\frac{\beta}{3}} \\ & \times D_\infty^{-1} \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 \end{pmatrix} (0 \ 0 \ 1) (I + \mathcal{O}(n^{-1/6})) \end{aligned} \quad (6.9)$$

Now let us now replace z and a by the scaled variables

$$z_n = \frac{z}{\sqrt{2}n^{3/2}}, \quad a_n = -1 + \frac{\sqrt{2}\tau}{n^{1/2}}. \quad (6.10)$$

It then follows from (6.10), and (4.74)–(4.75) that

$$n^{\frac{3}{2}}f(z_n; a_n) = z + \mathcal{O}(n^{-\frac{1}{2}}), \quad n^{\frac{1}{2}}\tau(z_n; a_n) = \tau + \mathcal{O}(n^{-\frac{1}{2}}), \quad (6.11)$$

as $n \rightarrow \infty$, where the \mathcal{O} -terms hold uniformly for z in a bounded set. Then we obtain from (6.9)

$$\begin{aligned} R_0(z_n)E_n(z_n; a_n) &= \frac{Ce^{c_1+c_2} e^{-\frac{1}{6}\tau^2}}{\sqrt{6\pi}} \frac{2^{\frac{1}{2}\beta}}{3^{\frac{1}{2}(\alpha+\gamma)+\beta}} n^{\frac{1}{2}-\frac{\beta}{2}} \\ & \times D_\infty^{-1} \begin{pmatrix} \sqrt{2}i \\ 1 \\ -1 \end{pmatrix} (0 \ 0 \ 1) (I + \mathcal{O}(n^{-1/6})) \end{aligned} \quad (6.12)$$

We insert (6.10), (6.11), and (6.12) into (6.4). We also note that D_∞ depends in an analytic way on a , so that by (4.38)

$$(D_\infty^{-1})_{1,1} = D_0(\infty; a_n)^{-1} = C^{-1}(1 + \mathcal{O}(n^{-1/2})).$$

Then we obtain

$$\begin{aligned} P_{n,n}(z_n; a_n) &= (-1)^n e^{-\frac{2n}{3}(l_1+l_2)} \frac{e^{c_1+c_2} e^{-\frac{1}{6}\tau^2}}{\sqrt{3\pi}} \frac{2^\beta}{3^{\frac{1}{2}(\alpha+\gamma)+\beta}} n^{\frac{1}{2}+\beta} \\ &\quad \times iz^{-\beta} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \Psi(z; \tau) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} (I + \mathcal{O}(n^{-1/6})) \end{aligned} \quad (6.13)$$

The constants l_1 and l_2 depend on a and therefore by (6.10) on τ . One can show that

$$e^{-\frac{2n}{3}(l_1+l_2)} = \left(\frac{4}{27}\right)^n e^{-\sqrt{2}\tau n^{\frac{1}{2}}} e^{-\frac{5\tau^2}{6}} \cdot \left(1 + \mathcal{O}\left(n^{-\frac{1}{2}}\right)\right) \quad (6.14)$$

Thus we proved (1.10) where C_n is given by (1.14) and Q is

$$Q(z) := iz^{-\beta} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \Psi(z; \tau) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \quad (6.15)$$

Looking at the definition (2.14) of $\Psi(z; \tau)$ with $0 < \arg z < \frac{\pi}{4}$, we find from (6.15)

$$Q(z) = iz^{-\beta} (e^{2\beta\pi i} q_1''(z) + q_2''(z)) \quad (6.16)$$

Then by the integral representation (2.13) of q_1 and q_2 we find by easy contour deformation from (6.16), that for $\operatorname{Re} z > 0$,

$$Q(z) = iz^{-\beta} \int_{\Gamma_0} t^{-\beta-1} \exp\left(\frac{\tau}{t} - \frac{1}{2t^2} + zt\right) dt, \quad (6.17)$$

with Γ_0 as in Figure 2. Making the change of variable $zt \mapsto t$, we arrive at the integral in (1.11), which shows that Q from (6.15) indeed agrees with the Q defined by (1.11) in the theorem.

This completes the proof of Theorem 1.4.

Remark 6.1. The property (4.90) implies that $(I + Z_n(z; a))^{-1} = I - Z_n(z; a)$ and so by (4.65)

$$(I - Z_n(z; a))P_0(z)N(z)^{-1} = I + \mathcal{O}(n^{-1/6}) \quad \text{for } z \in \partial U_0.$$

This suggests an alternative approach in which we redefine $P_0(z)$ with the extra factor $I - Z_n(z; a)$ on the left. Using this redefined P_0 in the transformation (5.1) from $S \mapsto R_0$, we would arrive at a RH problem for R_0 that has all jumps close to the identity matrix, but which has a simple pole at $z = 0$. This can be included in the RH problem by providing a residue condition at $z = 0$. The simple form of

the residue matrix $Z_n^{(0)}(a)$ allows one to remove the pole again in a transformation $R_0 \mapsto R$ as in [18]. The resulting R is then exactly the same as before.

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